Towards Optimal Dynamic Graph Sparsification

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Department of Computer Sciences
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08.11.2017

A Definition

A graph $G = (V, E)$ consists of

- a set of n nodes V and
- a set of m edges
  $E \subseteq \{\{u, v\} \mid u, v \in V\}$.

Graphs model **binary relationships** between entities

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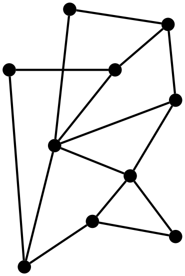

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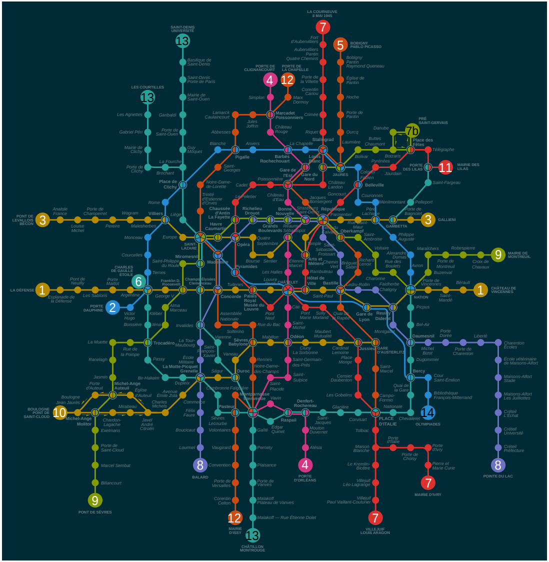
Graphs are Everywhere



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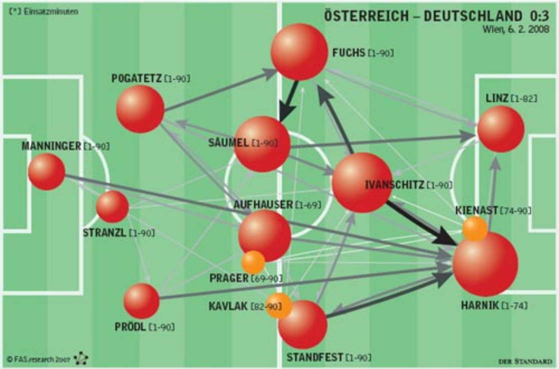

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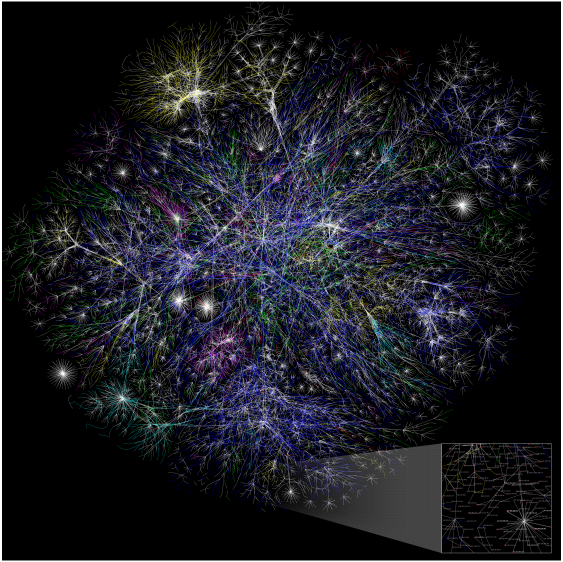
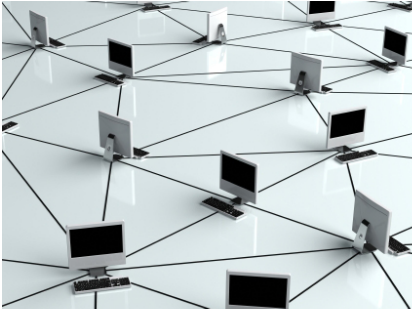
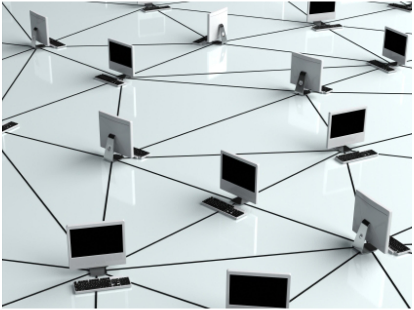


Graphs are Everywhere

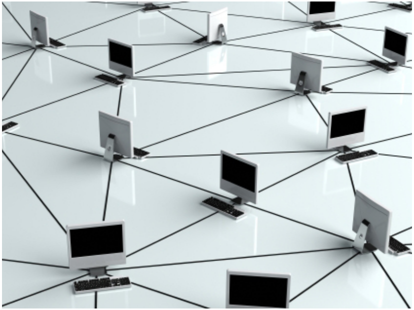
Research Area 1

Distributed and Parallel Algorithms






Shaw
KUKA
CITIZEN

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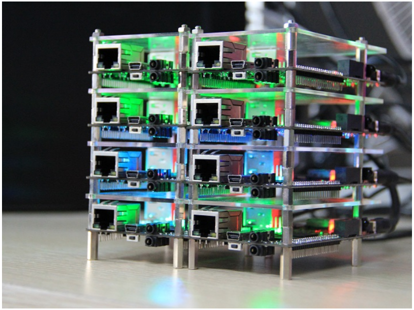
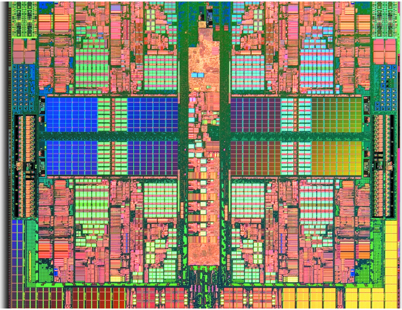




Shortest Path Algorithms

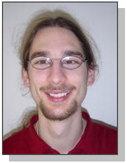
Single-Source Shortest Paths in distributed **CONGEST** model

- Improved exact algorithm
- **Close-to-optimal** approximation algorithm

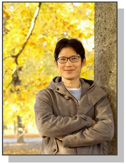
Ruben Becker

Monika Henzinger

Andreas Karrenbauer



Christoph Lenzen



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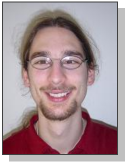
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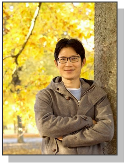
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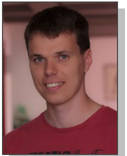
SSSP in parallel RAM model:

- Better parallelization in presence of negative edge weights
- Improves a **sequential** problem: minimum cost-to-time ratio cycle


Karl Bringmann


Thomas Dueholm Hansen

Research Area 2

Hardness of Polynomial-Time Problems

Complexity Theory for a Big-Data World

Conventional wisdom in complexity theory (70s-90s?):

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Can we rule out the existence of truly subquadratic time algorithms for certain problems?

Yes! …under plausible hardness assumptions

Conditional Lower Bounds

Fine-grained complexity of **diameter approximation**

- No subquadratic algorithm under Strong Exponential Time Hypothesis [Roditty/V. Williams '13]
- Not even subquadratic $\frac{3}{2}$-approximation
- Goal: more detailed hardness analysis

Karl Bringmann

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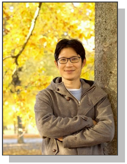
Karl Bringmann

Conditional lower bounds for **dynamic problems**

- Formulation of new hardness conjecture
- Explains certain barriers in dynamic algorithms

Monika Henzinger



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Thatchaphol Saranurak

Research Area 3

Dynamic Algorithms

Our World is not Static

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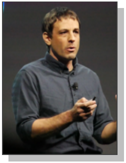
Goal: Fast recomputation of solution after update in the graph

Research on Dynamic Algorithms

Fastest dynamic **shortest path** algorithm in a variety of settings (7+ papers)



Ittai
Abraham

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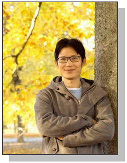

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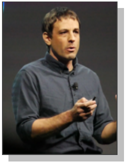

Ittai Abraham

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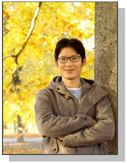

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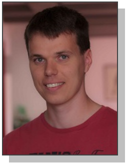
Dynamic **connectivity** and **dominators** in directed graphs

Loukas Georgiadis

Giuseppe Italiano



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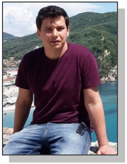

Nikos Parotsidis

Sparsification

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Idea: Approximate dense objects by sparse objects

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$\Longrightarrow$

Truss arch

Sparsification in Graphs

Goal: Reduce to much smaller set of edges

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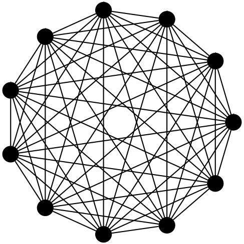
Dense graph



$m = \Omega(n^2)$

Sparsification in Graphs

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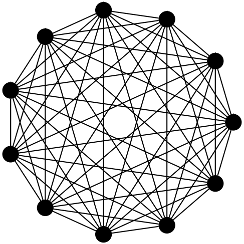

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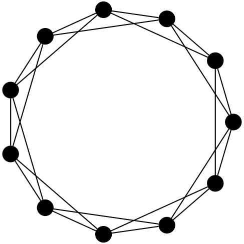

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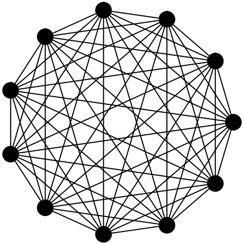

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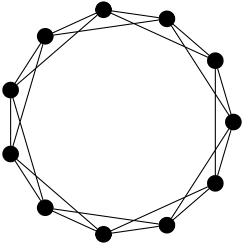

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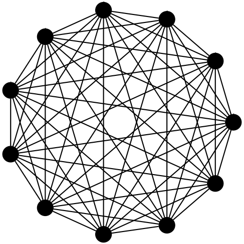

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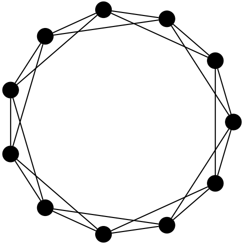

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At cost of approximation

Example 1: Distance Sparsifier

Definition

A **spanner** of **stretch** α of $G = (V, E)$ is a subgraph $H = (V, E')$ such that

$$dist_G(u, v) \leq dist_H(u, v) \leq \alpha \cdot dist_G(u, v)$$

for all pairs of nodes $u, v \in V$.

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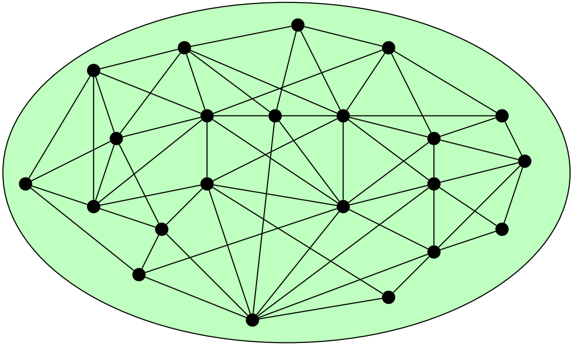

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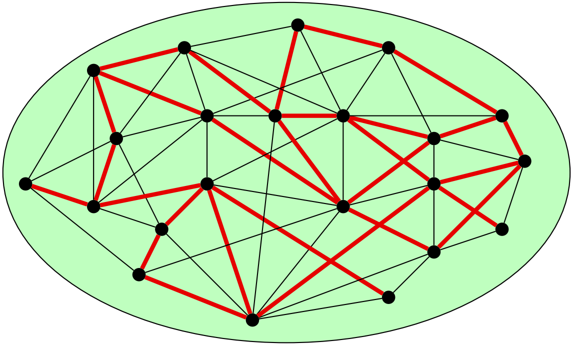

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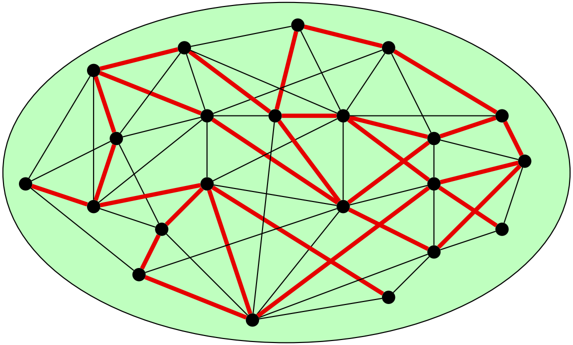
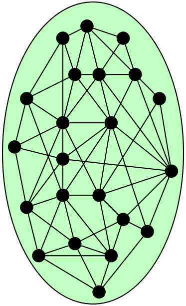


Fact: Every graph has spanners with stretch $(2k - 1)$ of size $n^{1+1/k}$ $(k \geq 2)$

In particular: stretch $\log n$ and size $O(n)$

Dynamic Problem

Input graph G



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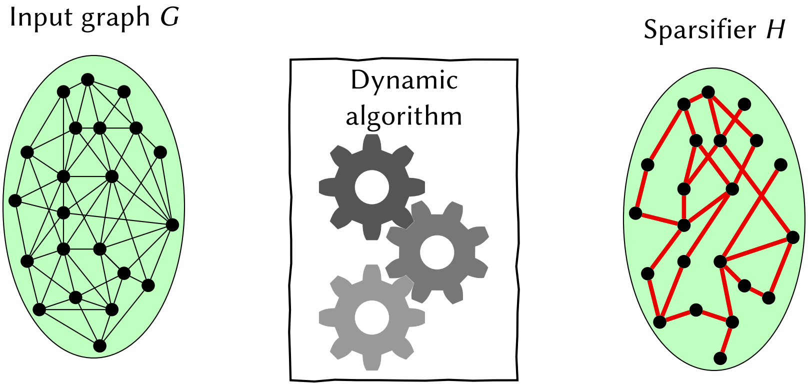

Dynamic Problem

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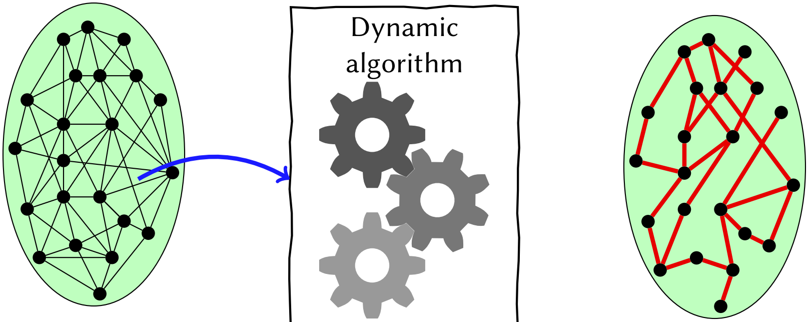
Sparsifier H



adversary inserts and deletes edges

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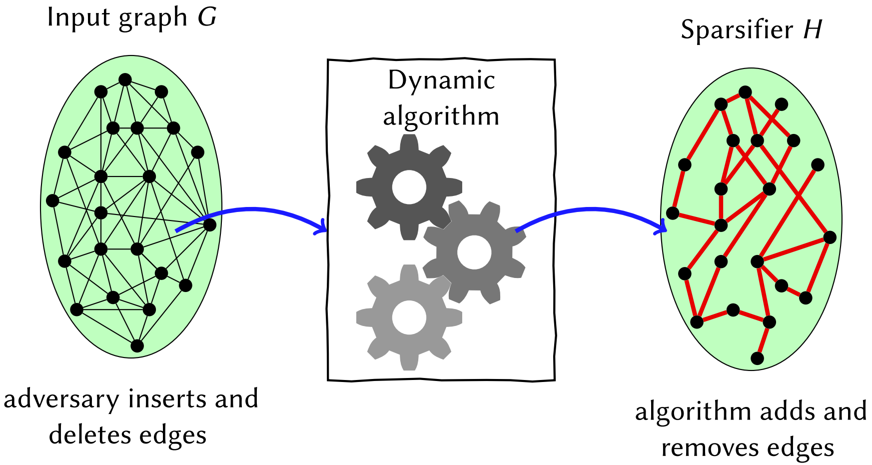

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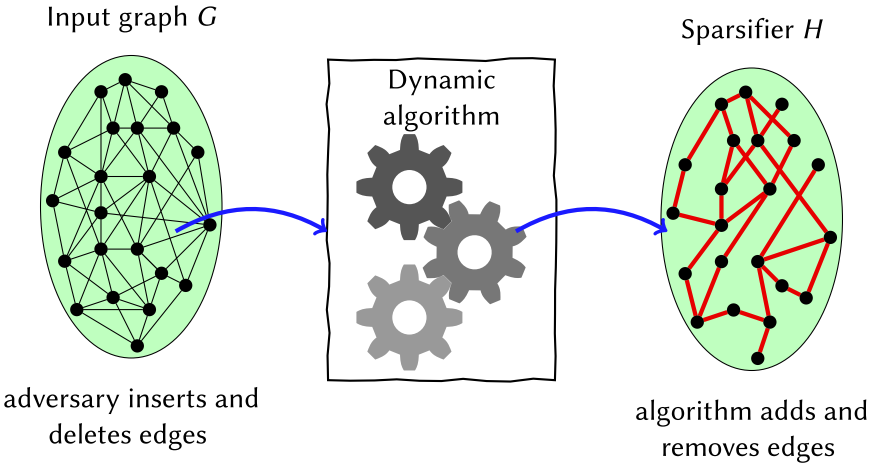

State of the art update time:

- Amortized time: $O(k^2 \log^2 n)$, stretch $2k - 1$
  Total time $O(t \cdot k^2 \log^2 n)$ for t updates [Baswana et al. 2012]

Dynamic Problem

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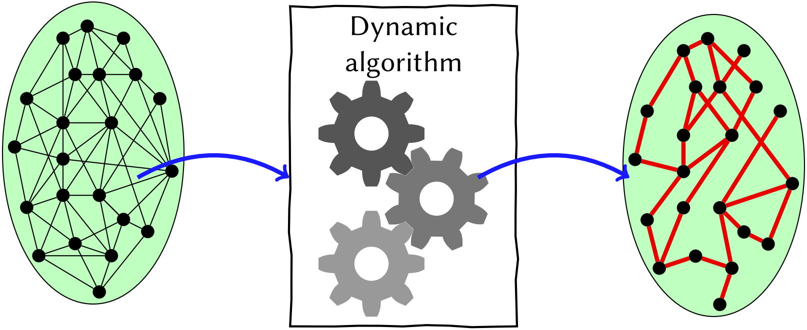

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algorithm adds and removes edges

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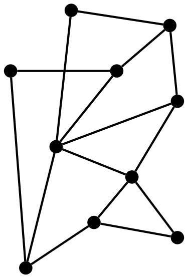
- Worst-case time: $O(n^{3/4})$ for stretch 3 [Bodwin/K 2016]

Greg Bodwin

Example 2: Spectral Sparsification

View graph G as Laplacian matrix L_G



$\longrightarrow$

$$\begin{pmatrix} 2 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 3 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & -1 & -1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 3 & -1 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & -1 & 4 & -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & -1 & 4 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 3 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 & 0 & -1 & 0 & -1 & 0 & 3 \end{pmatrix}$$

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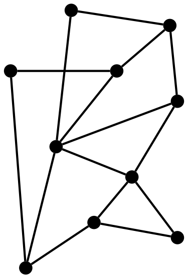

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$$L_G[i, i] = degree(v_i)$$

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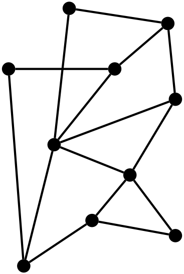

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$$(1-\varepsilon)x^T L_G x \leq x^T L_H x \leq (1+\varepsilon)x^T L_G x$$

for all vectors $x \in \mathbb{R}^n$.

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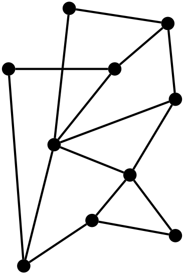

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Definition

A $(1 \pm \varepsilon)$-spectral sparsifier of G is a weighted subgraph H such that

$$(1-\varepsilon)x^T L_G x \le x^T L_H x \le (1+\varepsilon)x^T L_G x$$

for all vectors $x \in \mathbb{R}^n$.

Under Löwner ordering: $(1-\varepsilon)L_G \preceq L_H \preceq (1+\varepsilon)L_G$

Motivation I: Cut Sparsification

Consider set of nodes $S \subseteq V$ and vector $x \in \mathbb{R}^n$ such that

$$x_i = 1 \text{ if } i\text{-th node in } S$$
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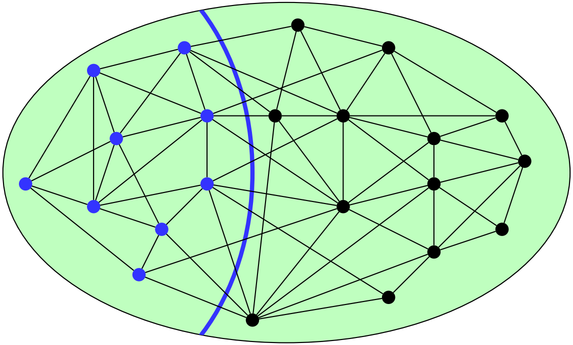

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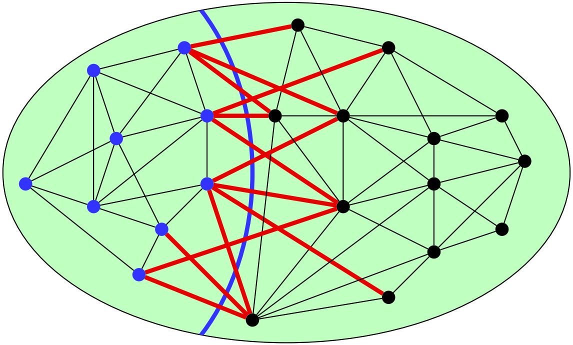

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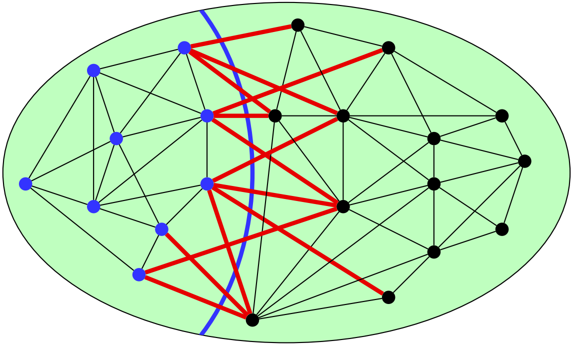

⇒ Spectral sparsifier is also a cut sparsifier [Benczúr/Karger '00]

Motivation II: Solving SDD Systems

System of linear equations with *n* unknowns:

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
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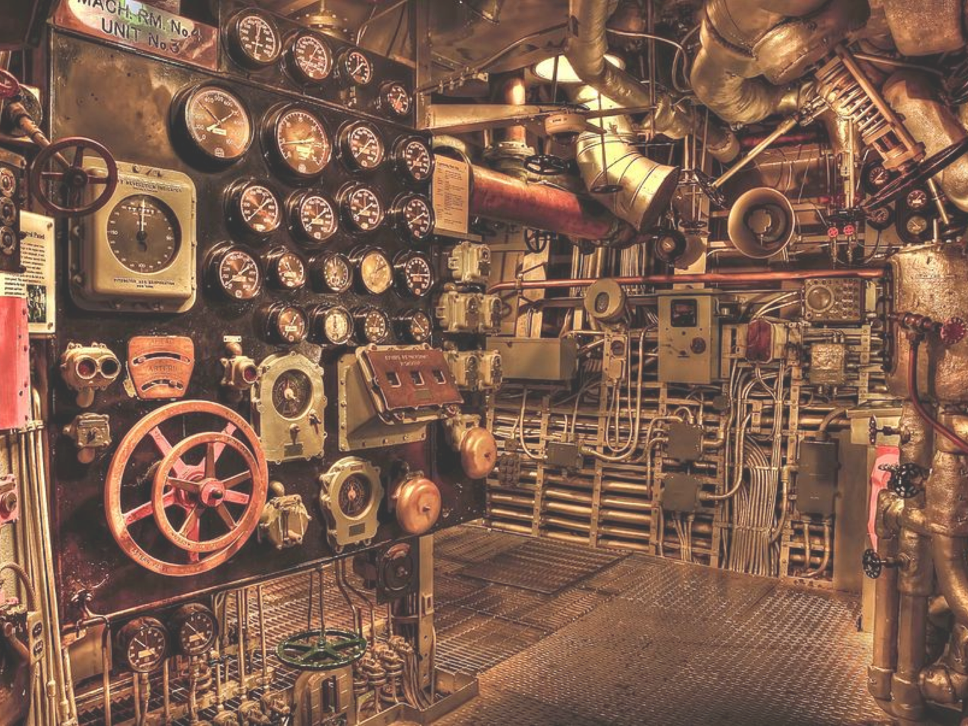
Dynamic Solver?

Changing one row in $A \rightarrow$ changing $\leq 2n$ edges of G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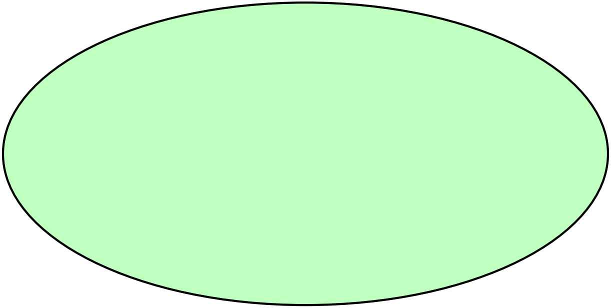
UNIT No.3

t-Bundle Spanners



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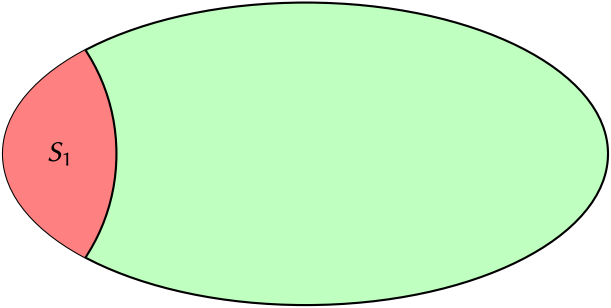

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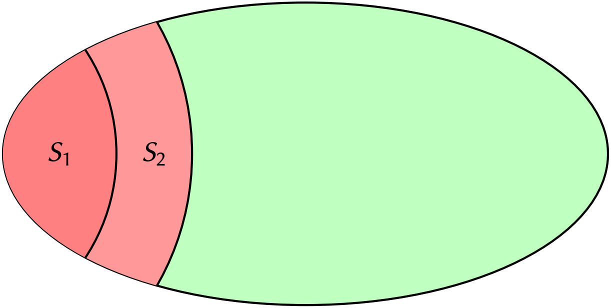

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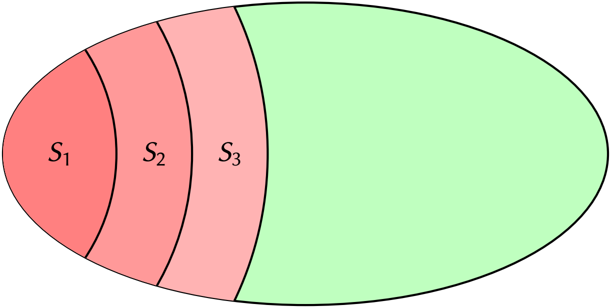

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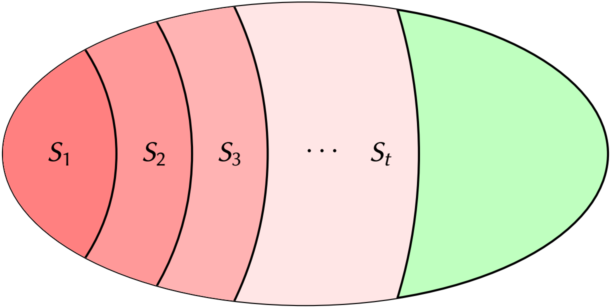

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  $\vdots$

- Compute spanner S_t of $G \setminus (S_1 \cup S_2 \cup \cdots \cup S_{t-1})$

$B := S_1 \cup S_2 \cup \cdots \cup S_{t-1}$ is a t-**bundle spanner**

1-Step Spectral Sparsification

1. Compute t-bundle spanner B with $t = \frac{24 \log^2 n}{\varepsilon^2}$
2. Set $H = B$
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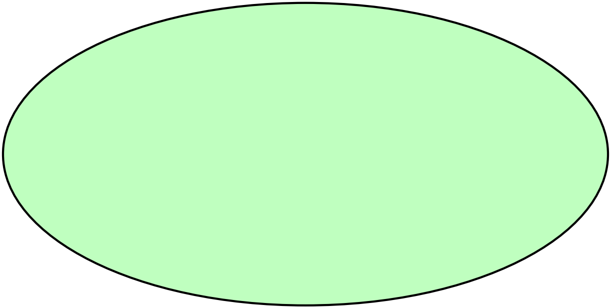
Repeat 1-step sparsification on remaining graph until size is small enough



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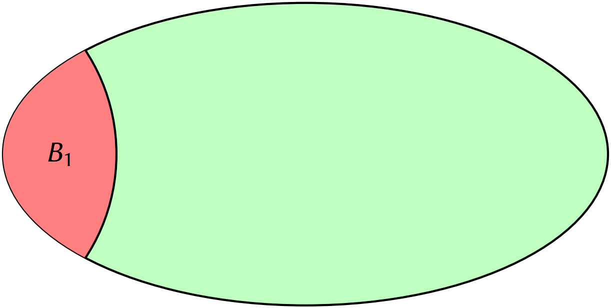

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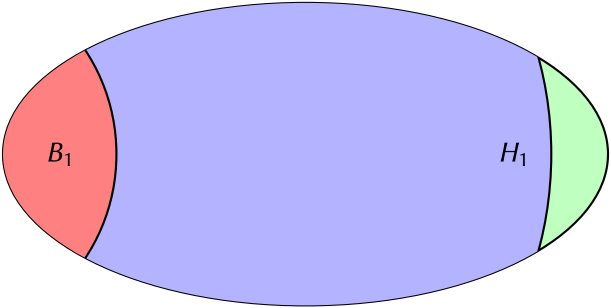

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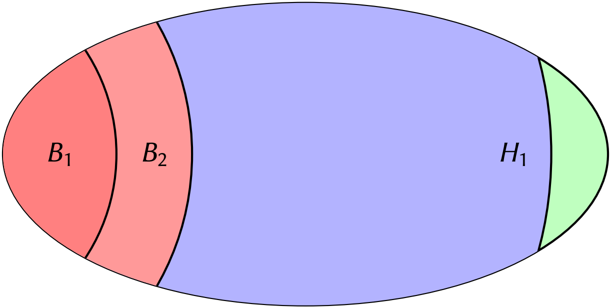

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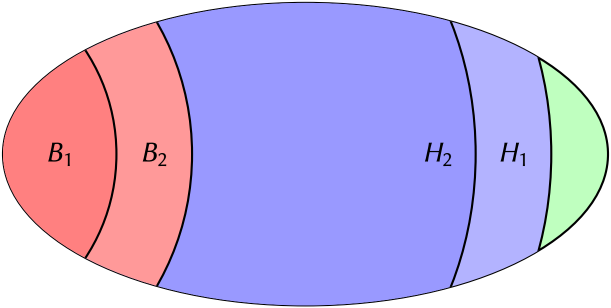

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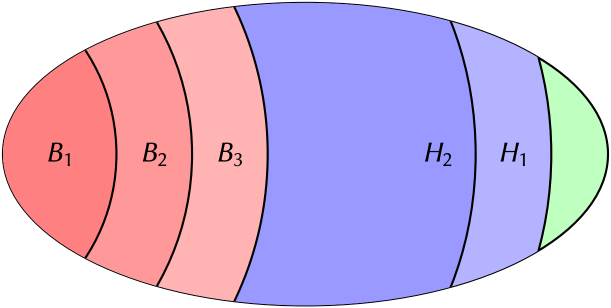

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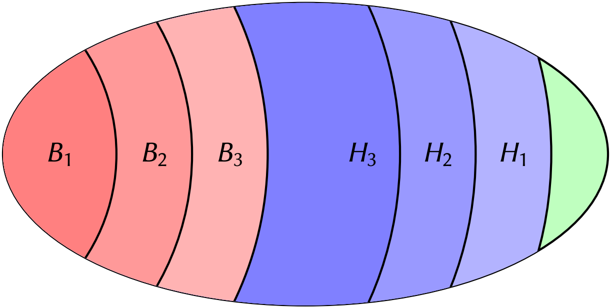

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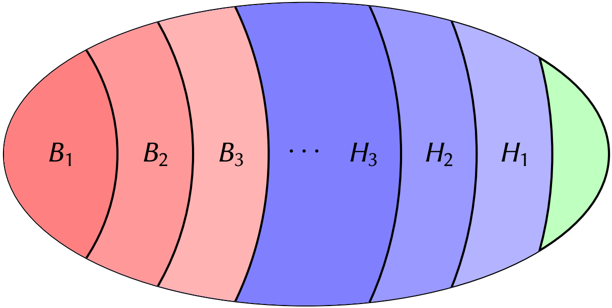

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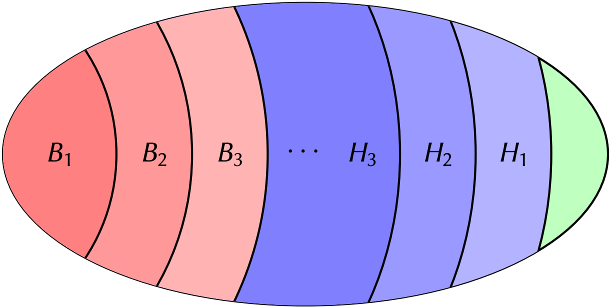

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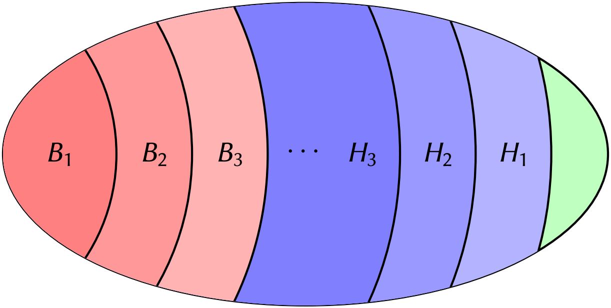

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Good parallelization due to parallel spanner algorithm [Baswana/Sen '03]

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Straightforward approach:

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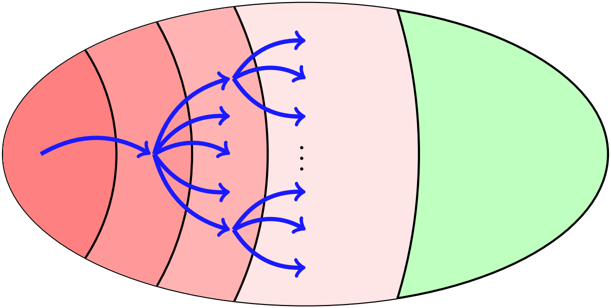

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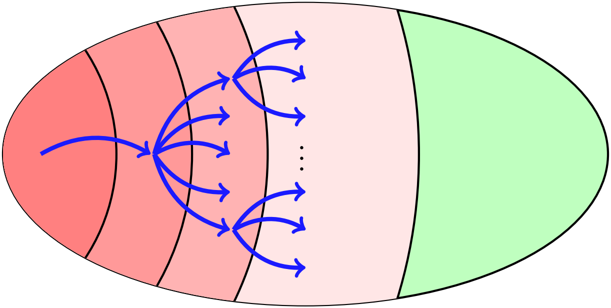
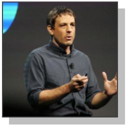


Solution: Refined algorithm design

Our Algorithm



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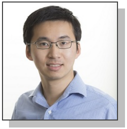

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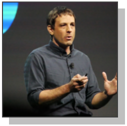
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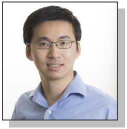
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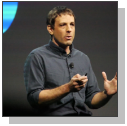

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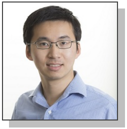

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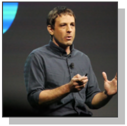

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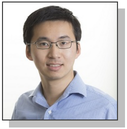

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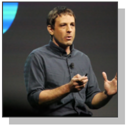
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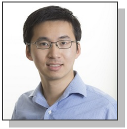
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Theorem (Abraham et al. '16)

There is a dynamic algorithm for maintaining a spectral sparsifier of size $n \cdot poly(\log n, \varepsilon^{-1})$ with amortized update time $poly(\log n, \varepsilon^{-1})$ per edge insertion/deletion.

Conclusion

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Questions?