Distributed Laplacian Solving with Applications

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Joint work with Gramoz Goranci, Yang P. Liu, Richard Peng, Xiaorui Sun, Tijn de Vos, and Mingquan Ye

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Laplacian Paradigm

- Laplacian systems
- Spectral sparsifiers
- Electrical flow
- Effective resistance
- Expander decompositions
- Continuous optimization
- Interior-point methods
- Gradient descent
- Preconditioning
- ...

Laplacian Paradigm and Distributed Computing

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Laplacian paradigm often yields inherently parallelizable algorithms

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Basic operation:

- Vector $\mathbf{x}$: each node represents a coordinate
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State of the art for (approximate) single-source shortest path, maximum flow, minimum-cost flow:
[Ghaffari, Karrenbauer, Kuhn, Lenzen, Patt-Shamir '15] [Becker, F, Karrenbauer, Lenzen '17] [Zuzic '21] [Anagnostides, Themis Gouleakis, Christoph Lenzen '21] [Zuzic, Goranci, Ye, Haeupler, Sun '22] [Rozhon, Grunau, Haeupler, Zuzic, Li '22]

Laplacian Systems

Goal

Solve linear system $\mathbf{L}\mathbf{x} = \mathbf{b}$ such that $\mathbf{L}$ is a **Laplacian matrix**.

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Definition

The **Laplacian matrix** $\mathbf{L}(G)$ of graph $G = (V, E, w)$ is defined by

$$\mathbf{L}(G)_{u,v} = \begin{cases} \sum_{(u,v')\in E} w_{u,v'} & \text{if } u = v, \\ -w_{u,v} & \text{otherwise.} \end{cases}$$

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High-precision solver: Approximation of solution $\mathbf{x}^*$ with $\mathbf{x}$ s.t.

$$\|\mathbf{x} - \mathbf{x}^*\|_{\mathbf{L}(G)} \leq \epsilon \|\mathbf{b}\|_{\mathbf{L}(G)} .$$

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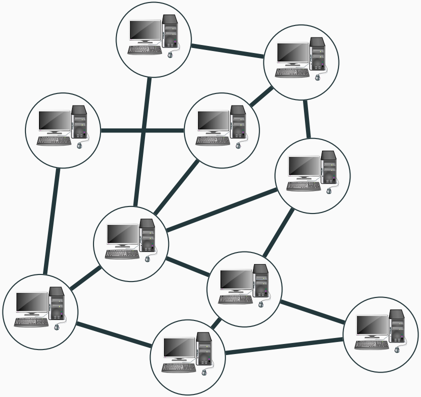
Prior work:

- $\tilde{O}(m)$ sequential running time [Spielman, Teng '04]
- $\tilde{O}(m)$ work, polylogarithmic depth [Peng, Spielman '14]

CONGEST Model



- Edges correspond to non-zero entries of matrix
- Each node holds one row/column of matrix
- Communication over edges in synchronous rounds
- Bandwidth $O(\log n)$ per edge

Our Results for the CONGEST Model

Theorem ([F, Goranci, Liu, Peng, Sun, Ye])

In the CONGEST model, given a weighted and undirected graph G and a vector $\mathbf{b}$ on n vertices, we can in $O(n^{o(1)}(\sqrt{n}+D))$ rounds return a vector $\mathbf{x}$ such that $\|\mathbf{x}-\mathbf{x}^*\|_{\mathbf{L}(G)} \leq \epsilon\|\mathbf{b}\|_{\mathbf{L}(G)}$.

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Implications

$\tilde{O}\left(m^{3/7+o(1)}(n^{1/2}D^{1/4}+D)\right)$-round algorithms in CONGEST model for the following problems:

- Maximum flow [Mądry '16]
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First $o(n)$-round algorithms for sparse, low-diameter graphs

Approximate Schur Complement

Definition (Schur complement)

For an $n \times n$ symmetric matrix $\mathbf{M}$ and a subset of *terminals* $T \subseteq [n]$, let $S = [n] \setminus T$. Permute the rows/columns of $\mathbf{M}$ to write

$$\mathbf{M} = \begin{bmatrix} \mathbf{M}_{[S,S]} & \mathbf{M}_{[S,T]} \\ \mathbf{M}_{[T,S]} & \mathbf{M}_{[T,T]} \end{bmatrix}.$$

Then the *Schur complement* of $\mathbf{M}$ onto T is defined as

$\mathrm{SC}(\mathbf{M}, T) := \mathbf{M}_{[T,T]} - \mathbf{M}_{[T,S]}\mathbf{M}_{[S,S]}^{-1}\mathbf{M}_{[S,T]}$.

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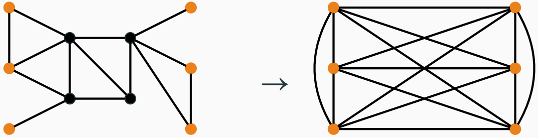

Input graph $\qquad$ Schur complement

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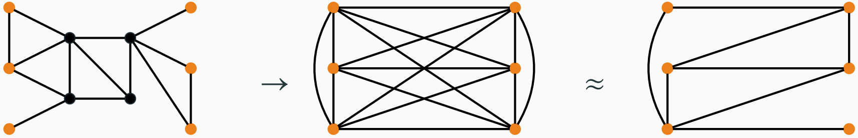

Input graph Schur complement Sparsification

Minor Sparsifiers

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Communication of edges "along" sparsifier edges my lead to too much congestion

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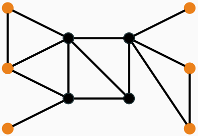
Vertex sparsifiers as **minors** of the communication graph



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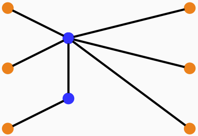

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Matrix-vector multiplication involving minor sparsifier takes $\tilde{O}(\sqrt{n} + D)$ rounds.

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Key contribution: Parallel variant of [Li Schild ’18]

Technical Details

- From [Kyng, Lee, Peng, Sachdeva, Spielman '16]: Repeated elimination of **almost independent sets** yields vertex sparsifier "chain" with recursion depth $d = O(\log n)$

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  - Distortion of minor property in recursive calls

What Next?

Implications

$\tilde{O}\left(m^{3/7+o(1)}(n^{1/2}D^{1/4} + D)\right)$-round algorithms in CONGEST model for the following problems:

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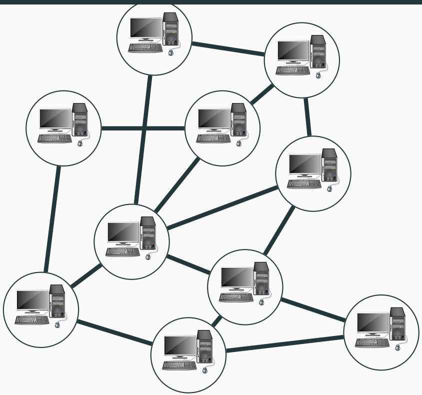
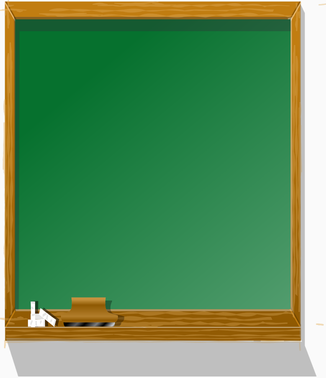
Easier Question

Sublinear #rounds on the Broadcast Congested Clique?

Broadcast Congested Clique





- Nodes can communicate with all other nodes [Lotker et al. ’05]
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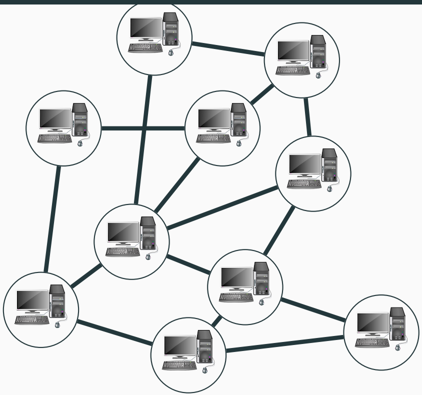
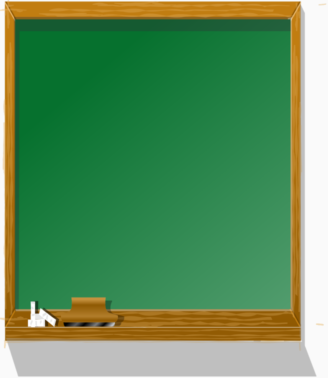
Broadcast Congested Clique





- Nodes can communicate with all other nodes [Lotker et al. ’05]
- Broadcast *the same* message to all nodes [Drucker, Kuhn, Oshman ’12]
- For many problems: only “trivialization” of CONGEST model upper bounds with $D = 1$ is known

Our Results for the BCC

Theorem ([F, de Vos '22])

On the Broadcast Congested Clique, the minimum cost flow problem can be solved in $\tilde{O}(\sqrt{n})$ rounds.

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Other Results:

- On the Broadcast Congested Clique, a spectral sparsifier of quality $1 \pm \epsilon$ and size $\tilde{O}(n/\epsilon^2)$ can be computed in $\tilde{O}(1/\epsilon^2)$ rounds
- On the Broadcast Congested Clique, a Laplacian system can be solved up to high accuracy in $\tilde{O}(\log^2(1/\epsilon))$ rounds
- On the Broadcast Congested Clique, certain Linear Programs can be solved in $\tilde{O}(\sqrt{n})$ rounds.

Main Idea and Challenges

Linear Programming

Minimize $\mathbf{c}^\top \mathbf{x}$ subject to $\mathbf{A}\mathbf{x} = \mathbf{b}$

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Minimum cost flow:

- Rank = #nodes
- Linear system has Laplacian matrix

Key Contribution

Iterative computation of spectral sparsifier [Koutis, Xu '16]:

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Solution:

- Compute spanner on "probabilistic" graph
- Sample individual edges ad-hoc when needed
- Modification of spanner algorithm of [Baswana, Sen '07]

Optimization vs. Data Structures

[Lee, Sidford '14]:
#iterations: $\tilde{O}(\sqrt{n})$
Time per iteration: $\tilde{O}(m)$

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- Better upper bound already interesting for single-source reachability

Conclusion

Almost optimal Laplacian solvers

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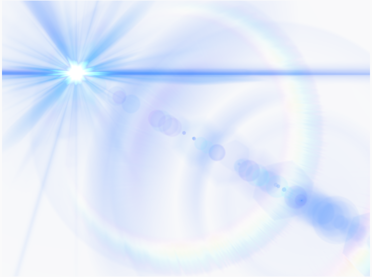

Almost optimal Laplacian solvers

Broadcast Congested Clique is an interesting "burning glass"