A Faster Distributed Single-Source Shortest Paths Algorithm

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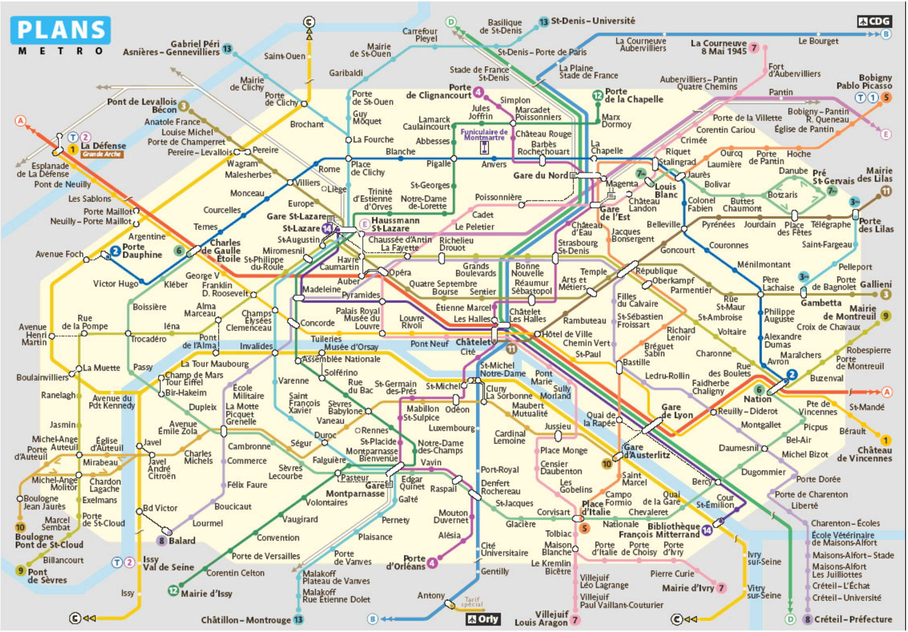
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Aubervilliers – Pantin Quatre Chemins
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Marcadet Poissonniers
Marx Dormoy
Bobigny – Pantin R. Queneau
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Brochant
Lamarck Caulaincourt
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Château Rouge
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Barbès Rochechouart
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Crimée
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Blanche
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Rome
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Stalingrad
Laumière
Pont de Neuilly
Malesherbes
Villiers
Gare du Nord
Jaurès
Danube
Pré St-Gervais
Mairie des Lilas
Liège
Trinité d'Estienne d'Orves
St-Georges
Notre-Dame de-Lorette
Magenta
Louis Blanc
Bolivar
Les Sablons
Monceau
Poissonnière
Château Landon
Colonel Fabien
Buttes Chaumont
Botzaris
Porte Maillot
Europe
Cadet
Gare de l'Est
Neuilly – Porte Maillot
Courcelles
Gare St-Lazare
Haussmann St-Lazare
Le Peletier
Château d'Eau
Belleville
Pyrénées
Jourdain
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Télégraphe
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Saint-Fargeau
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Miromesnil
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Havre Caumartin
Ménilmontant
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Grands Boulevards
Bonne Nouvelle
Temple
République
George V
Opéra
Père Lachaise
Porte de Bagnolet
Gallieni
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Kléber
Franklin D. Roosevelt
Auber
Quatre Septembre
Réaumur Sébastopol
Arts et Métiers
Oberkampf
Parmentier
Madeleine
Pyramides
Bourse
Sentier
Filles du Calvaire
Rue St-Maur
Gambetta
Boissière
Alma Marceau
Champs Élysées Clemenceau
Palais Royal Musée du Louvre
Étienne Marcel
Châtelet Les Halles
St-Ambroise
Philippe Auguste
Mairie de Montreuil
Avenue Henri Martin
Rue de la Pompe
Iéna
Concorde
Louvre Rivoli
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St-Sébastien Froissart
Rambuteau
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Trocadéro
Pont de l'Alma
Tuileries
Hôtel de Ville
Richard Lenoir
Voltaire
Alexandre Dumas
Invalides
Musée d'Orsay
Pont Neuf
Châtelet
Cité
Chemin Vert
Bréguet Sabin
Charonne
Robespierre
La Muette
Passy
La Tour Maubourg
Assemblée Nationale
St-Paul
Bastille
Maraîchers
Avron
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Boulainvilliers
Champ de Mars Tour Eiffel
St-Michel Notre-Dame
Pont Marie
Rue des Boulets
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Ledru-Rollin
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École Militaire
Varenne
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Félix Faure
Gare Montparnasse
Raspail
Denfert Rochereau
Les Gobelins
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Campo Formio
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Cour St-Émilion
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Porte de Versailles
Porte de Vanves
Porte d'Orléans
Ivry sur-Seine
Maisons-Alfort Les Juilliottes
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Corentin Celton
Malakoff Plateau de Vanves
Gentilly
Le Kremlin Bicêtre
Villejuif Léo Lagrange
Pierre Curie
Créteil – L'Échat
Issy
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Malakoff Rue Étienne Dolet
Antony
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Villejuif Paul Vaillant-Couturier
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- Not fully understood in CONGEST model
- To be fair: non-negative weights also not fully understood in RAM model

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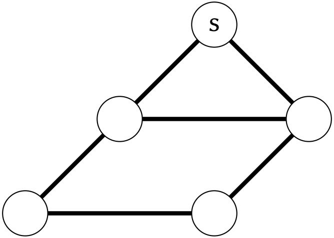
Distributed problem statement:

- Initial knowledge: incident edges, source
- Terminal knowledge: distance to the source, parent on shortest path tree

Unweighted Graphs: BFS



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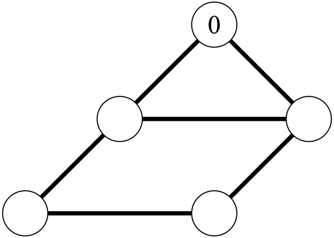

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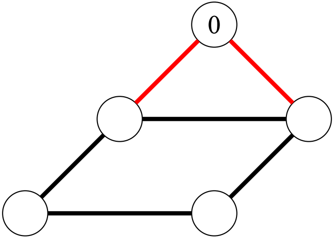

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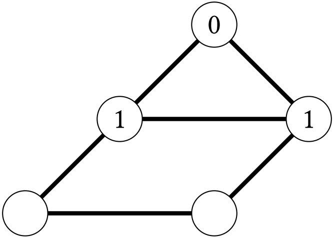

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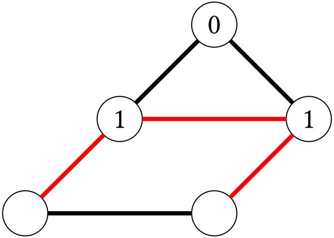

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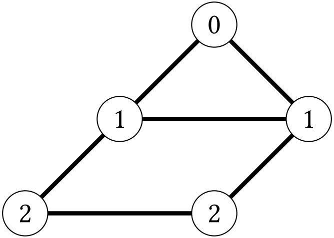

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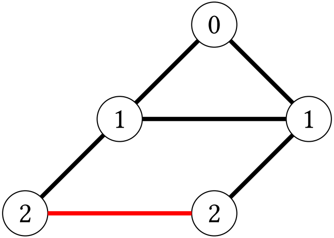

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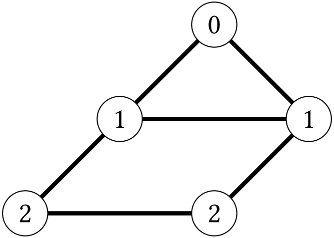

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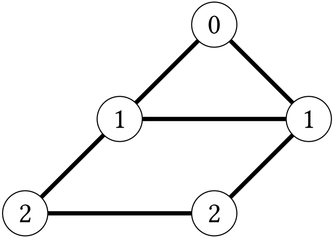

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Our goal: efficient algorithms for weighted graphs

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Lower Bound: $\tilde{\Omega}(\sqrt{n} + D)$ [Peleg/Rubinovich '99] [Das Sarma et al. '11]

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Approximation Algorithms: [Nanongkai '14] [Holzer and Pinsker '15] [Henzinger/K/Nanongkai '16] [Elkin/Neiman '16] [Becker et al. '17]

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Two scaling techniques [Gabow '85]:

1. **Bitwise scaling:** In each iteration read next bit of weights
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- Inherent dependence on $\log(W_{\max})$ due to maximum distance

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Theorem ([Klein/Subramanian '97])

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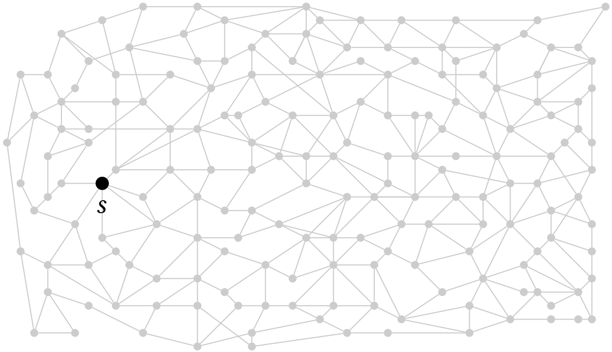
- Careful design to satisfy domination constraint

Omitted in this talk:

- Detailed running time analyis
- Dealing with 0-weight edges: Reduce to positive edge weights
- Faster approximation algorithm for directed graphs

Skeleton Graph



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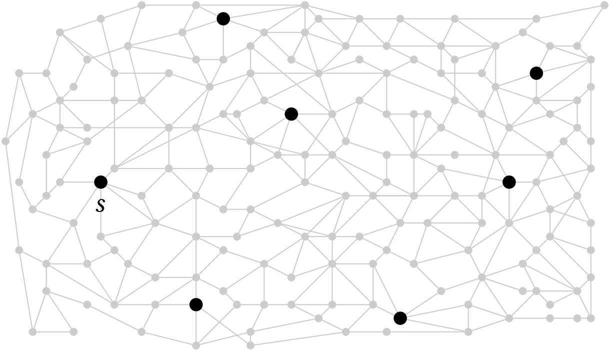

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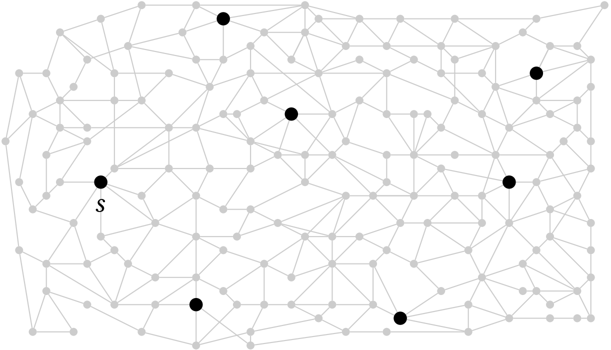

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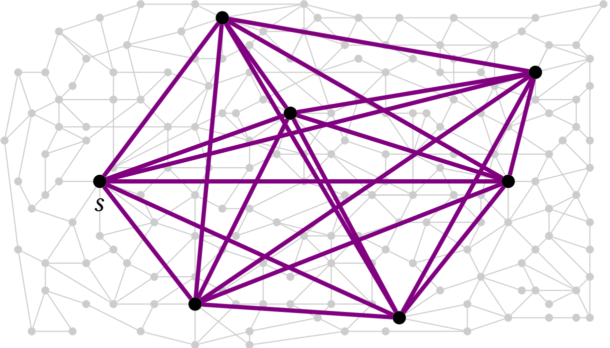

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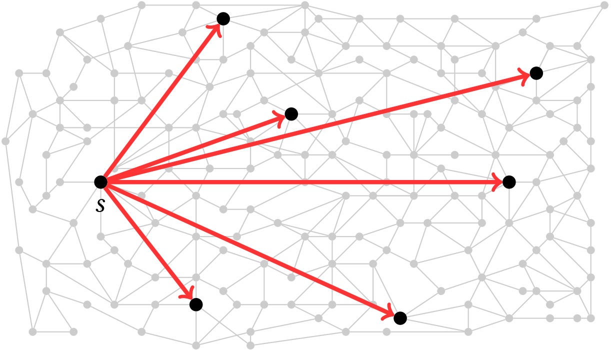

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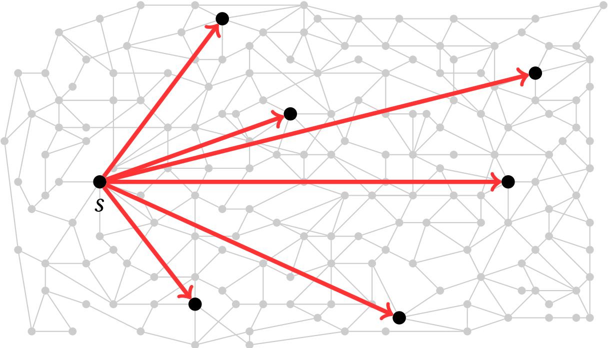

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Theorem

- *For every node v: $\frac{1}{2} \cdot \mathrm{dist}_G(s, v) \leq \hat{d}(s, v) \leq \mathrm{dist}_G(s, v)$ (approximation)*
- *For every edge (u, v): $\hat{d}(s, v) \leq \hat{d}(s, u) + w_G(u, v)$ (domination)*

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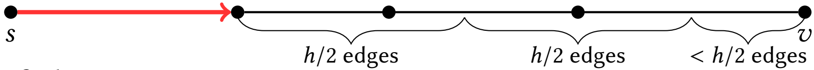

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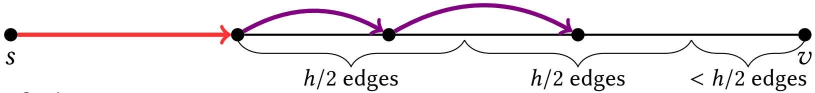

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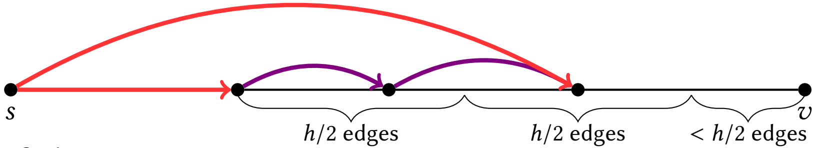

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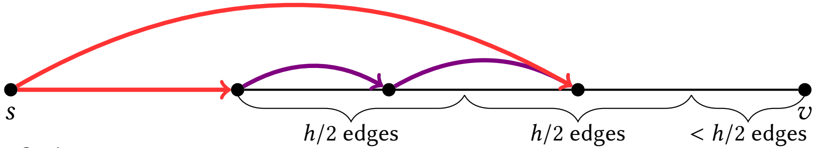

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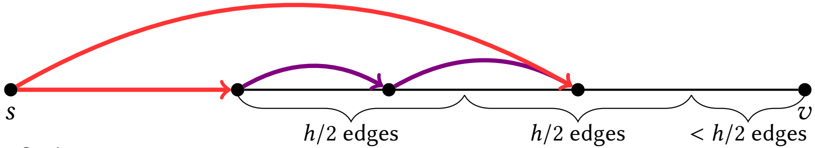

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New trade-off for directed graphs in PRAM model:

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Thank you for your attention!

slides: https://www.cosy.sbg.ac.at/~forster/