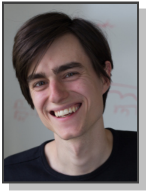
Fully Dynamic Spanners with Worst-Case Update Time Guarantees

Greg Bodwin
Stanford University



Sebastian Krinninger
Max Planck Institute for Informatics

European Symposium on Algorithms 2016

Motivation

Computing on Sparser Graphs

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Amortized vs. **Worst-Case** Bounds

- Many dynamic algorithms amortize running time over sequence of updates
- Not suitable for real-time systems: Hard guarantees needed

Spanners

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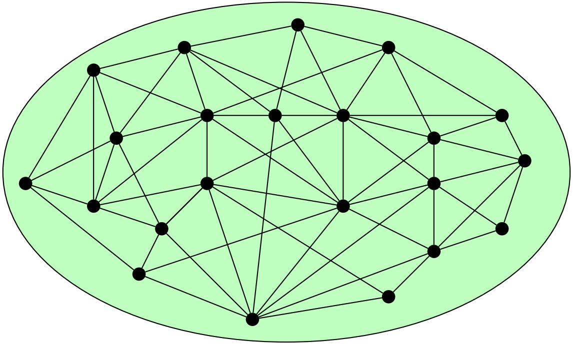

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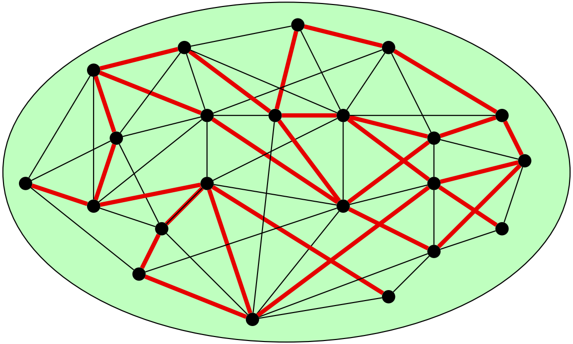

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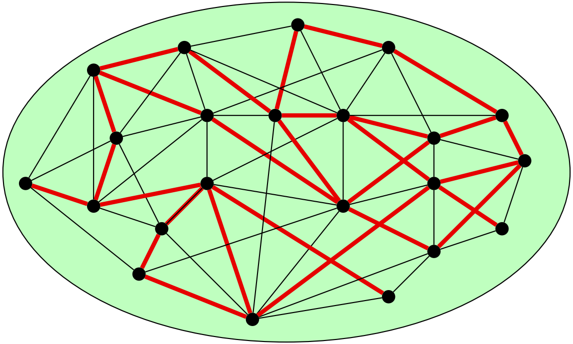
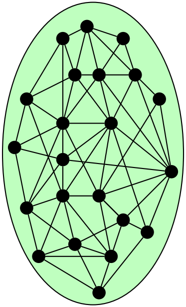


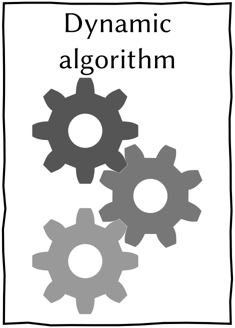
Fact: Every graph has a $(2k-1)$-spanner of size $n^{1+1/k}$ $(k \geq 2)$ [Folklore]

Essentially tight if **girth conjecture** is true [Erdős]

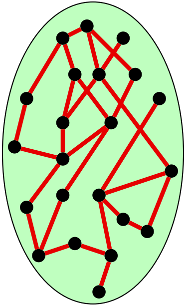
Dynamic Problem

undirected G





spanner H



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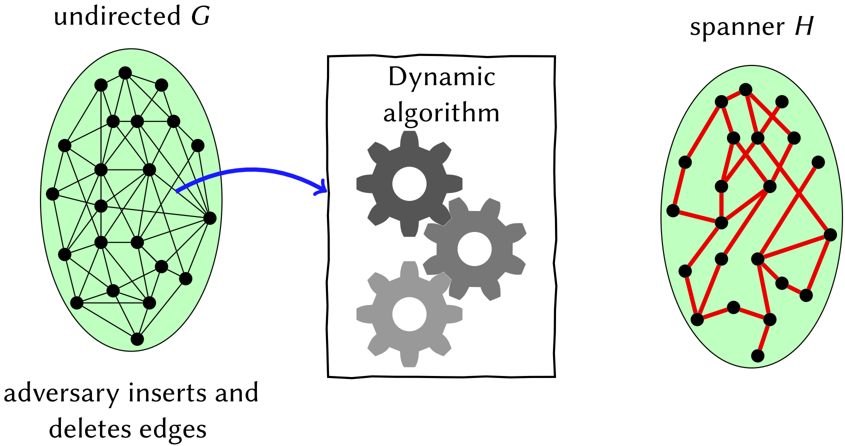

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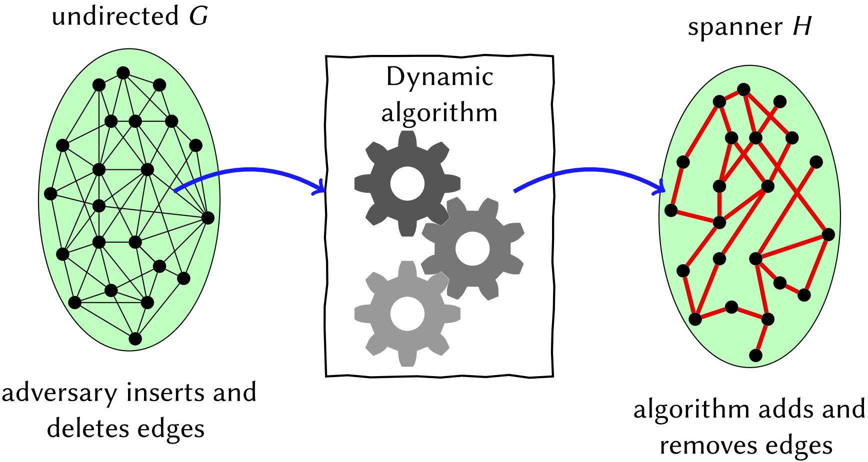

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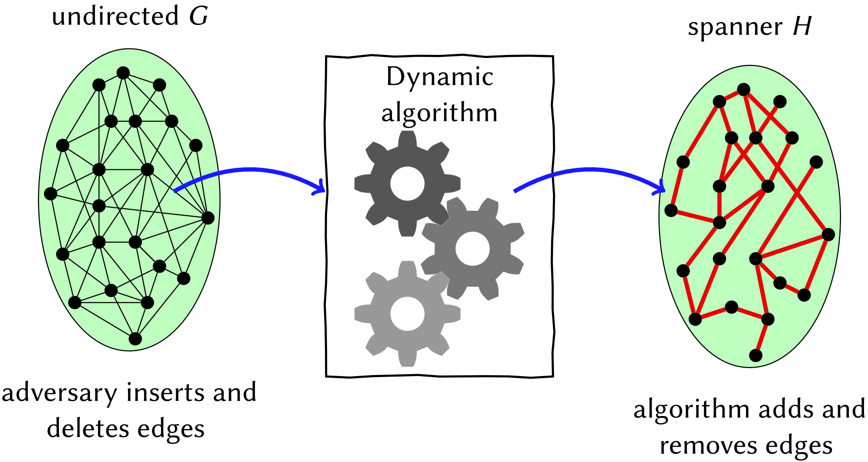

Goal: Maintain edges of spanner H with small update time after edge insertion/deletion in G

Our Results and Related Work

Amortized bounds:

stretch	*size*	*time*	*reference*

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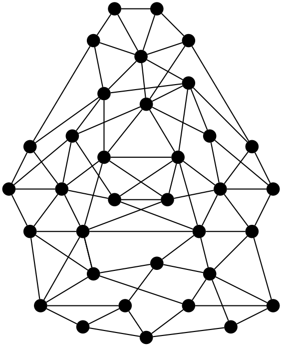
Will only show: stretch 3 in worst-case update time $O(n^{5/6})$

Spanner by Randomized Clustering



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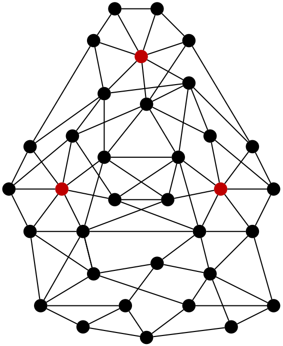

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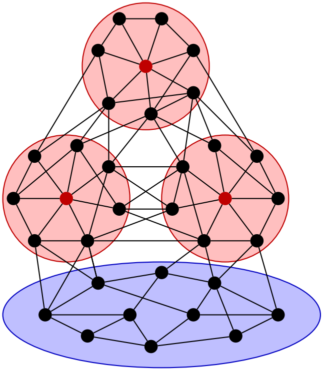

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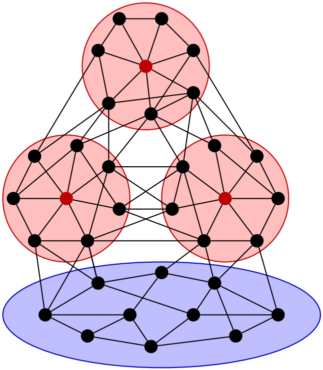

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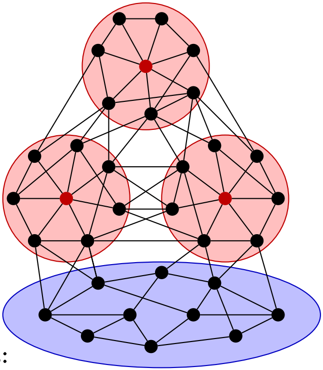

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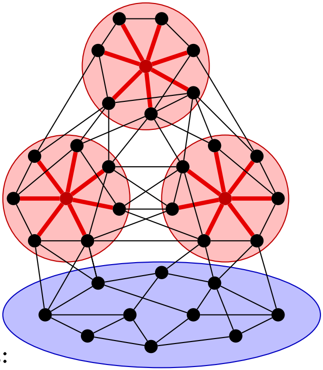

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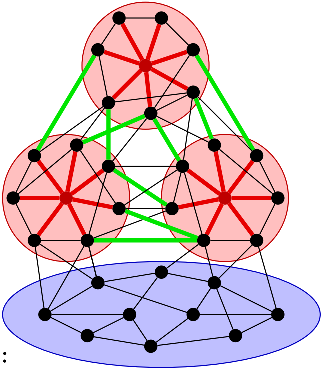

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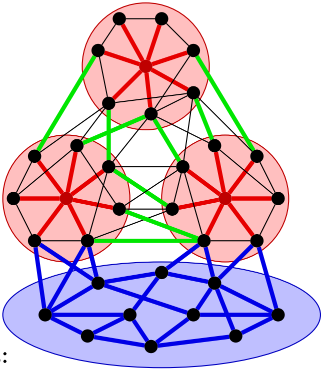

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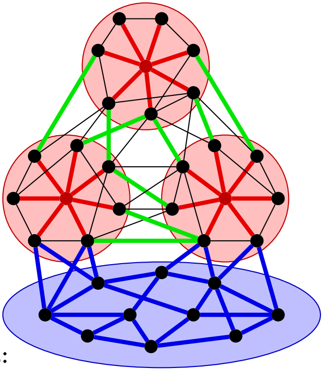

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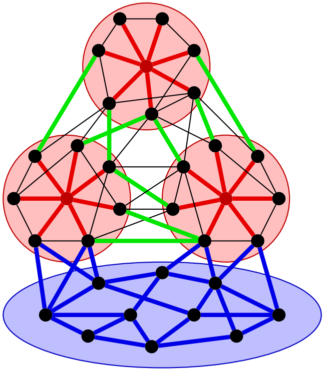
$\Rightarrow$ Spanner has stretch 3 and size $O(n^{1+1/2}\log n)$ whp (standard proof)

Maintaining Spanner I

- Random choice of centers at initialization



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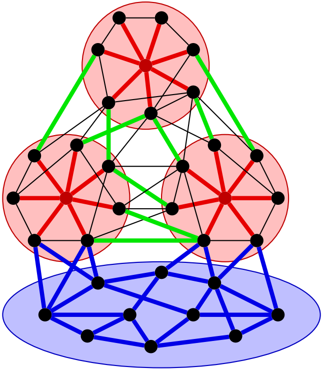

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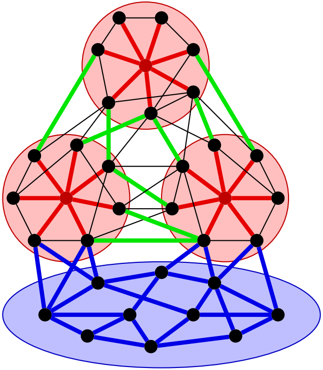

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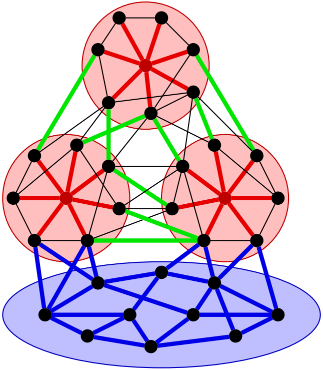

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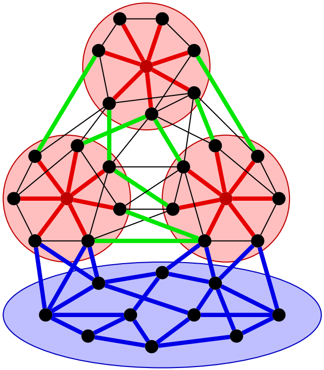

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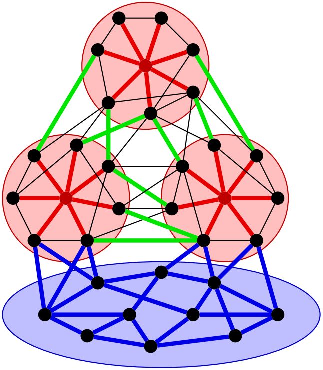
$\Rightarrow$ Update time: $O(maxdeg(G) \log n)$



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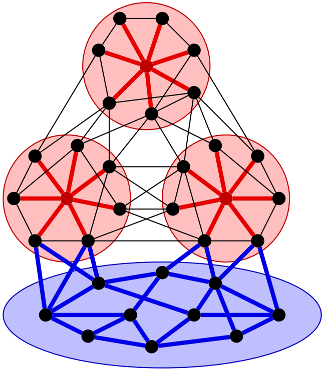

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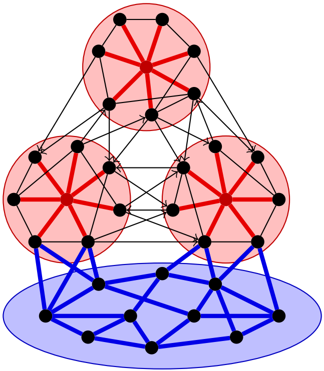

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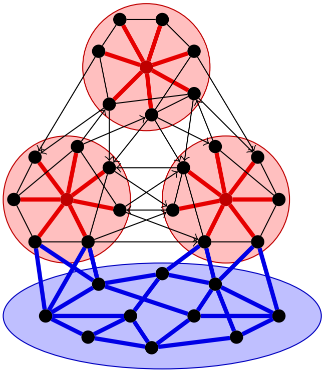

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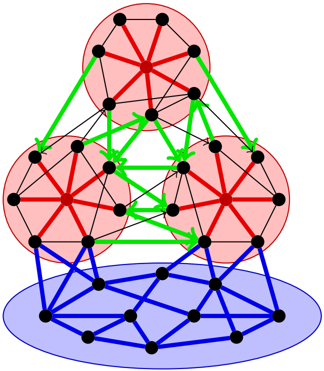

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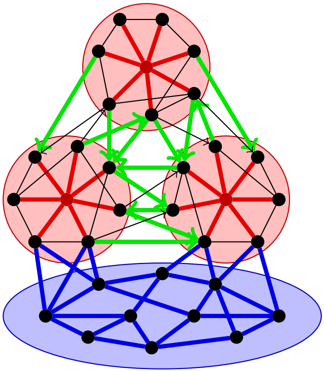
- For every clustered node v and every other cluster C, maintain set $In(v, C)$: incoming edges from cluster C to v
- Keep **one** entry of $In(v, C)$ in spanner
- No connection between clusters lost! For inter-cluster edge, one endpoint responsible to connect clusters



Maintaining Spanner II

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  Remove u from $N(v, i)$

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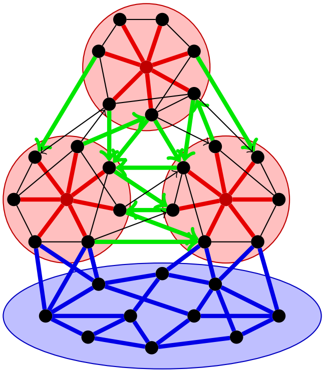

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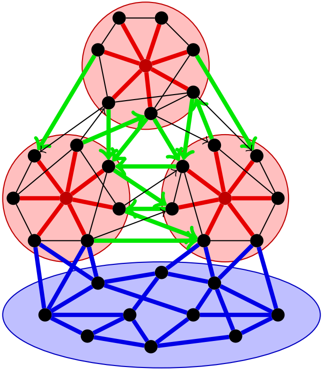
$\Rightarrow$ Update time: $O(maxoutdeg(\vec{G}) \log n)$



Partitioning Trick

Idea: Partition outgoing edges each node into groups of size s

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undirected graph G

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undirected graph	G
orient edges	$\downarrow$
	$\vec{G}$

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$$
\begin{array}{llllll}
\text{undirected graph} & & & G & & \\
\text{orient edges} & & & \downarrow & & \\
 & & & \vec{G} & & \\
\text{partition into subgraphs} & \swarrow & \swarrow & \dots & \searrow & \searrow \\
 & \vec{G}_1 & \vec{G}_2 & \dots & \vec{G}_{t-1} & \vec{G}_t
\end{array}
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\text{spanner} & & & H & &
\end{array}$$

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undirected graph			G		
orient edges			$\downarrow$		
			$\vec{G}$		
partition into subgraphs	$\swarrow$	$\swarrow$	$\dots$	$\searrow$	$\searrow$
	$\vec{G}_1$	$\vec{G}_2$	$\dots$	$\vec{G}_{t-1}$	$\vec{G}_t$
maintain sub-spanners	$\downarrow$	$\downarrow$	$\dots$	$\downarrow$	$\downarrow$
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Key observation

Each edge update has to be performed in only **one** subgraph

Update time: $O(maxoutdeg(\vec{G}_i)) = O(s)$

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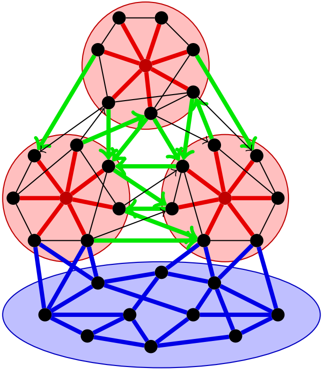
Smaller Spanner Size

Hierarchical approach:

- Clustering with $O((n \log n)/d)$ centers



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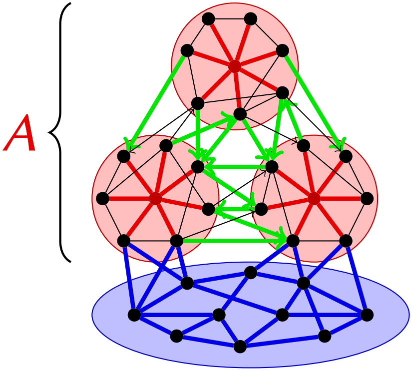

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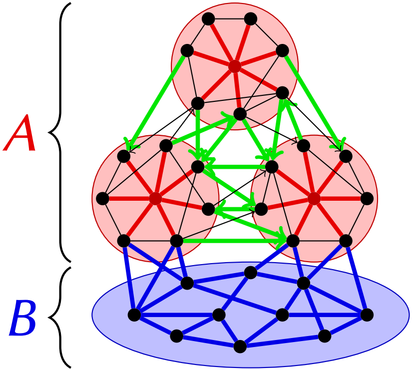

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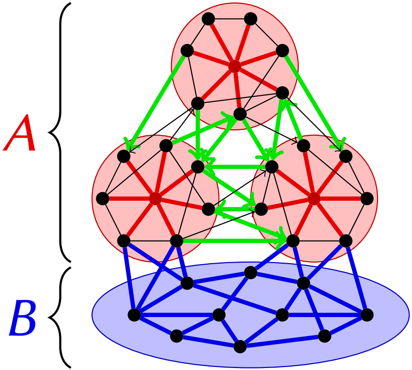
- $|A| \leq O((n^2 \log n)/d)$
- Every node in B has degree $\leq d$



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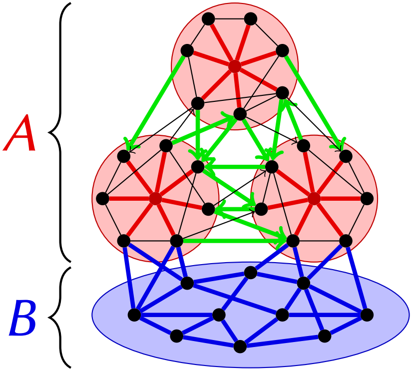

Observation: With every update in G, at most 4 edges are added to or removed from in H

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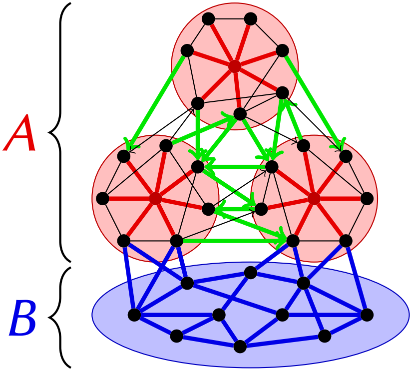

Observation: With every update in G, at most 4 edges are added to or removed from in H

- **Every** node has edges to its **first** d neighbors in spanner
- When node becomes **unclustered**, incident edges already contained

Full Algorithm

undirected graph G

Full Algorithm

undirected graph $\quad G$

orient edges $\quad \downarrow$

$\vec{G}$

Full Algorithm

undirected graph	G
orient edges	$\downarrow$
	$\vec{G}$
partition into subgraphs	$\swarrow \quad \dots \quad \searrow$
	$\vec{G}_1 \qquad \dots \qquad \vec{G}_t$

Full Algorithm

$$\begin{array}{llllllll}
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\text{maintain partitioned sub-spanners} & \swarrow & & \searrow & \dots & \swarrow & & \searrow \\
 & A_1 & & B_1 & \dots & A_t & & B_t
\end{array}$$

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undirected graph	G
orient edges	$\downarrow$
	$\vec{G}$
partition into subgraphs	$\swarrow \quad \dots \quad \searrow$
	$\vec{G}_1 \qquad \dots \qquad \vec{G}_t$
maintain partitioned sub-spanners	$\swarrow \qquad \searrow \quad \dots \quad \swarrow \qquad \searrow$
	$A_1 \qquad B_1 \quad \dots \quad A_t \qquad B_t$
union of unclustered parts	$\searrow \quad \dots \quad \swarrow$
	B

Full Algorithm

undirected graph	G
orient edges	$\downarrow$
	$\vec{G}$
partition into subgraphs	$\swarrow \quad \dots \quad \searrow$
	$\vec{G}_1 \quad \dots \quad \vec{G}_t$
maintain partitioned sub-spanners	$\swarrow \quad \searrow \quad \dots \quad \swarrow \quad \searrow$
	$A_1 \quad B_1 \quad \dots \quad A_t \quad B_t$
union of unclustered parts	$\searrow \quad \dots \quad \swarrow$
	B
maintain spanner	$\downarrow$
	H'

Full Algorithm

undirected graph				G			
orient edges				$\downarrow$			
				$\vec{G}$			
partition into subgraphs			$\swarrow$	$\dots$	$\searrow$		
		$\vec{G}_1$		$\dots$		$\vec{G}_t$	
maintain partitioned sub-spanners	$\swarrow$		$\searrow$	$\dots$	$\swarrow$		$\searrow$
	A_1		B_1	$\dots$	A_t		B_t
union of unclustered parts			$\searrow$	$\dots$		$\swarrow$	
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Final spanner: $H = A_1 \cup \cdots \cup A_t \cup H'$

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$s = n^{5/6}$, $d = n^{2/3}$, logarithms omitted

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Size of spanner: $O(t \cdot n^2/d + n^{1+1/2}) = O(n^3/(sd) + n^{1+1/2}) = O(n^{1+1/2})$

$s = n^{5/6}$, $d = n^{2/3}$, logarithms omitted

Conclusion

Summary:

- Main idea: Orienting and partitioning edges
- Careful hierarchy unleashes full potential

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Summary:

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Open Problems:

- Emerging barrier of $\sqrt{n}$: lower bound?
- Worst-case update time for larger stretches
- Sublinear deterministic algorithms

Questions?