Single-Source Shortest Paths: Towards Optimality

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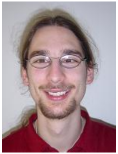
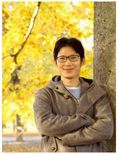
ADGA 2018

joint works with

				
Ruben Becker	Monika Henzinger	Andreas Karrenbauer	Christoph Lenzen	Danupon Nanongkai

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- To be fair: non-negative weights also not fully understood in RAM model

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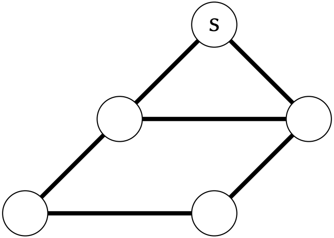
Distributed problem statement:

- Initial knowledge: incident edges, source
- Terminal knowledge: distance to the source, parent on shortest path tree

Unweighted Graphs: BFS



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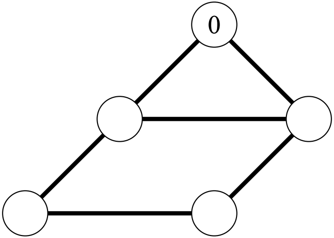

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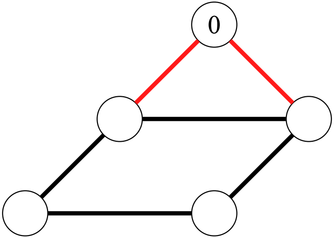

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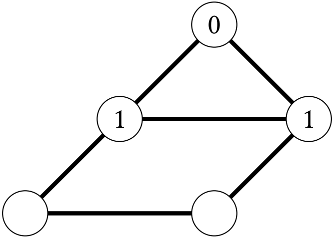

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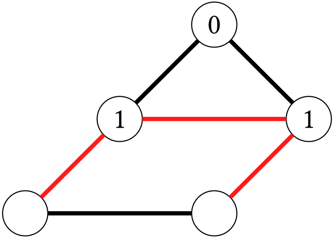

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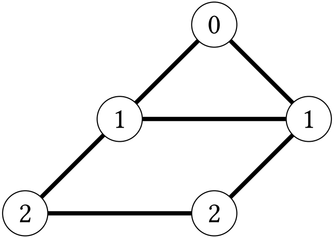

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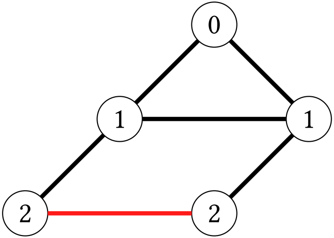

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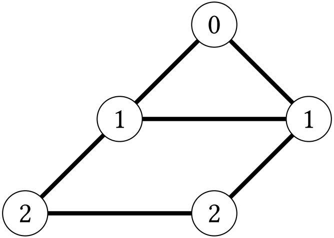

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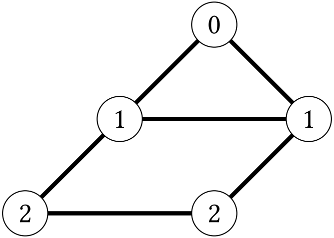

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Our goal: efficient algorithms for weighted graphs

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Exact SSSP:

$O(n)$	Bellman-Ford
$\tilde{O}(n^{2/3}D^{1/3} + n^{5/6})$	[Elkin '17]

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Common Lower Bound:

$\tilde{\Omega}(\sqrt{n} + D)$ — [Peleg/Rubinovich '99] [Elkin '04] [Das Sarma et al. '11]

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More Related Work

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- [Holzer/Wattenhofer '12]
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Congested Clique:

- [Censor-Hillel et al. ’15]
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Basic Tools

Broadcasting

Lemma

Suppose k pieces of information (of size $O(\log n)$ each) are distributed among the nodes of the network. All this information can be made known to all nodes in $O(k + D)$ rounds.

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Algorithm:

1. Initialize $\delta_0(s) = 0$ and $\delta(v) \neq$ for $v \neq s$
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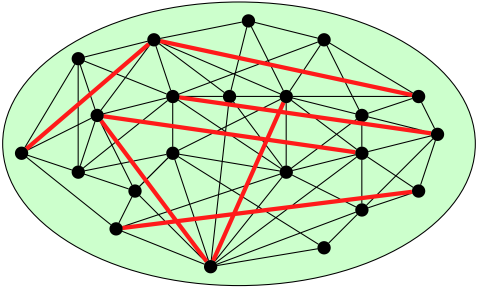
Intuition

SSSP is easy if shortest path has only few edges (hops)!

Hopsets

Definition ([Cohen '00])

An (h, ϵ)-hopset is a set of weighted edges F such that, for every pair of nodes u and v, there is a path from u to v with *at most h edges* of weight at most $(1+\epsilon)\,\mathrm{dist}_G(u,v)$ in $G \cup F$.



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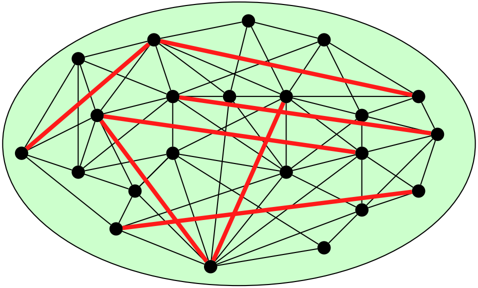

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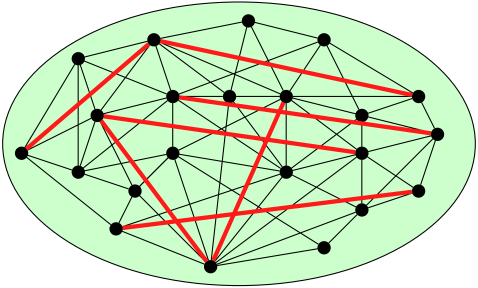

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Attention: Hopset edges cannot literally be "added" to network!

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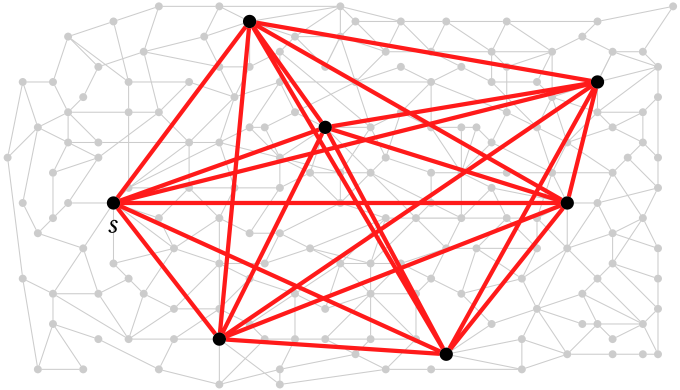

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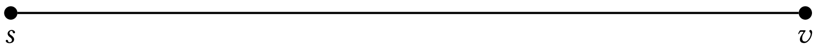

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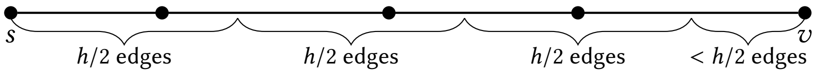

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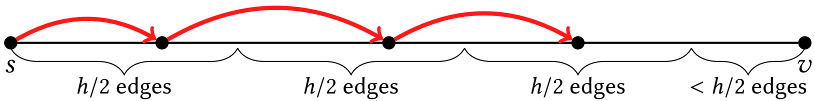

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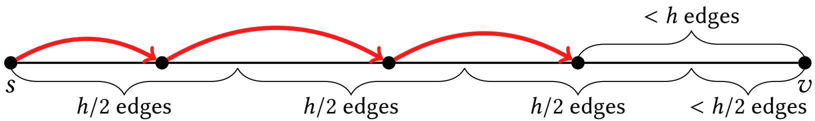
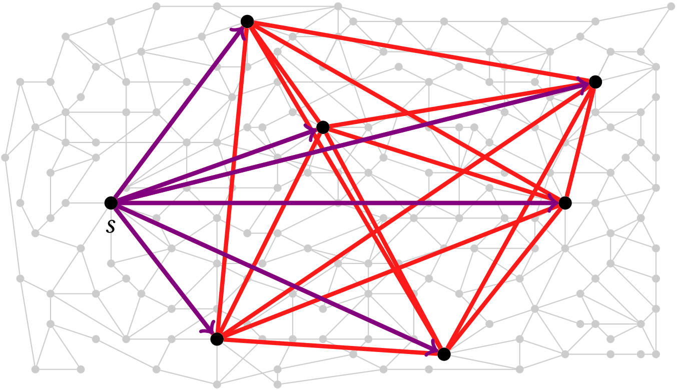


Skeleton Shortcuts: Intuition



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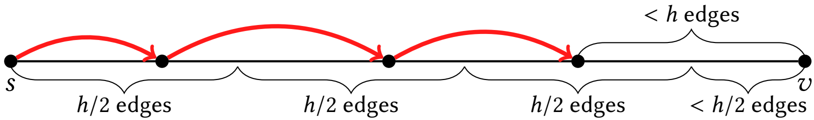

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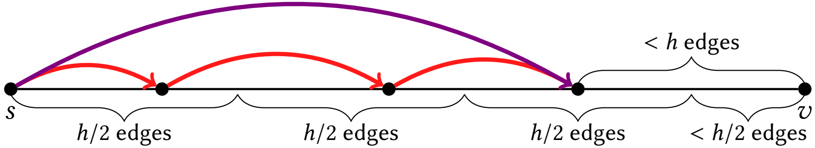

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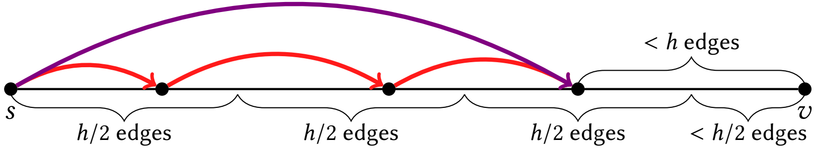

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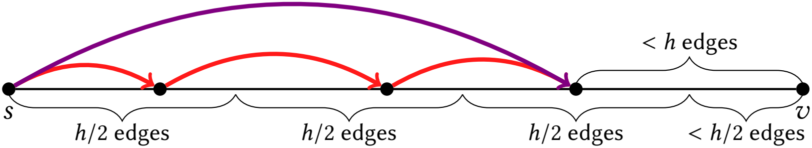

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- Only first iteration uses shortcut edges of F
- If each skeleton node knows shortcut to s, simulate first iteration in $O(D)$ rounds

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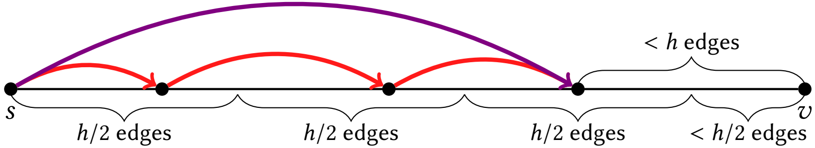

Good news:

- Cannot literally "add" shortcuts to network, but can run Bellman-Ford on $G \cup F$
- Only first iteration uses shortcut edges of F
- If each skeleton node knows shortcut to s, simulate first iteration in $O(D)$ rounds $\rightarrow O(h + D)$ rounds

A First Idea

Algorithm 1:

1. Determine skeleton nodes: random sample of $\tilde{O}(n/h)$ nodes + s (repeat sampling if too large)
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- Can compute shortest paths of weight $\leq L$ in time $O(L)$
- Bandwidth-friendly: at most one message per node
- Pseudopolynomial: h-hop shortest paths in time $O(hW_{\max})$

Approximate Bounded-Hop Distances

Weight rounding technique: [Klein/Subramanian '97]

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- But: Each edge traversal gives additive error of φ
- Choice of $\varphi_i = \epsilon 2^i / h$ deals with range $2^i \leq \text{dist}^h(s, v) \leq 2^{i+1}$

Lemma ([Nanongkai '14])

Can compute $(1+\epsilon)$-approximate h-hop shortest paths from given source in $\tilde{O}(h/\epsilon)$ rounds such that each node sends $\tilde{O}(1/\epsilon)$ messages

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- Approximate skeleton is $(\tilde{O}(n/h + h), \epsilon)$ hopset

Refined Algorithm

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Theorem

Can compute $(1+\epsilon)$*-approximate SSSP in time* $\tilde{O}(n^{2/3}/\epsilon + D)$ *with* $h = n^{2/3}$

Computing on Skeleton via Broadcast

Goal: Recurse on skeleton to improve efficiency

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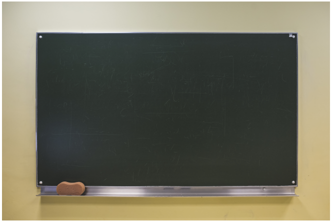
Idea: Simulate a round with total of k messages on skeleton by making all messages global knowledge in time $O(k + D)$

Reduction to Blackboard model

Blackboard model:

- Communication in synchronized rounds
- Write messages on "blackboard" to make them global knowledge
- No congestion constraint, only total size of messages is relevant



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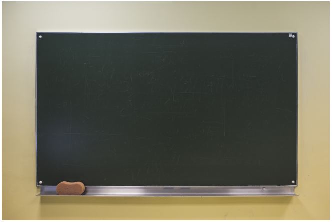

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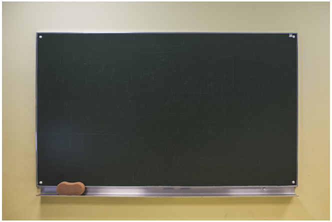

Lemma ([Nanongkai '14])

Any algorithm with $R(k)$ rounds and messages of total size $M(k)$ in blackboard model, can be simulated on skeleton of k nodes in $\tilde{O}(M(k) + R(k)D)$ rounds in the CONGEST model.

Back to Our Algorithm

Algorithm 3:

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Theorem ([F/Nanongkai '18])

Can compute $(1+\epsilon)$*-approximate SSSP in time* $\tilde{O}((\sqrt{n}D^{1/4} + D)/\epsilon)$ *with* $h = \sqrt{n}D^{1/4}$ *and* $h' = \sqrt{n}/D^{3/4}$

Exact SSSP

Scaling Approach

Two scaling techniques [Gabow '85]:

1. **Bitwise scaling:** In each iteration read next bit of weights
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Theorem ([Klein/Subramanian '97])

Suppose auxiliary algorithm computes distance estimate $\hat{d}(s, \cdot)$ such that

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  $\rightarrow$ Reduction to positive edge weights

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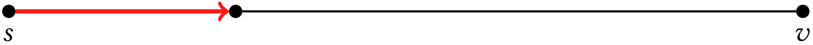

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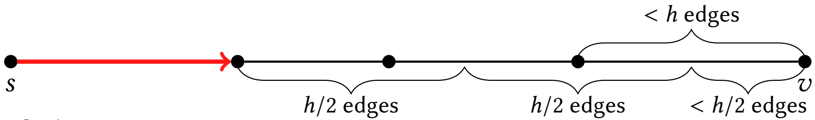

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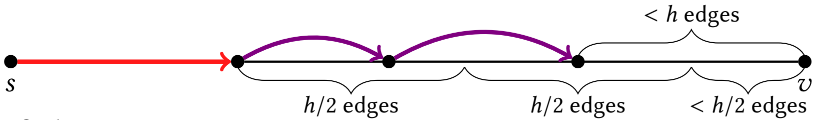

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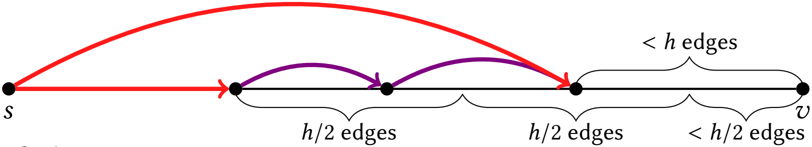

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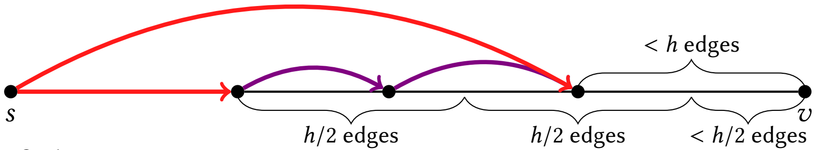

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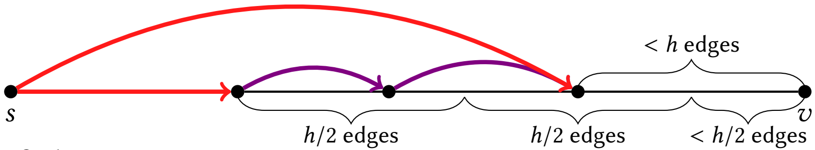

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Recall: We need *exact* SSSP on skeleton to compute shortcuts

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New trade-off for directed graphs in PRAM model:

- Klein and Subramanian: work $\tilde{O}(m\sqrt{n})$ and depth $\tilde{O}(\sqrt{n})$
- Our approach: work $\tilde{O}((n^3/h^3 + mh + mn/h))$ and depth $\tilde{O}(h)$

≈

Faster Approximation

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Model:

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Remarks:

- Hopset lower bound indicates $n^{o(1)}$ barrier [Abboud/Bodwin/Pettie '17]
- Tight hopsets exist [Huang/Pettie '17] [Elkin/Neiman '17]

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3. Is $\tilde{O}(\sqrt{n})$ rounds tight on *Broadcast* Congested Clique?

Thank you!

slides: https://www.cosy.sbg.ac.at/~forster/