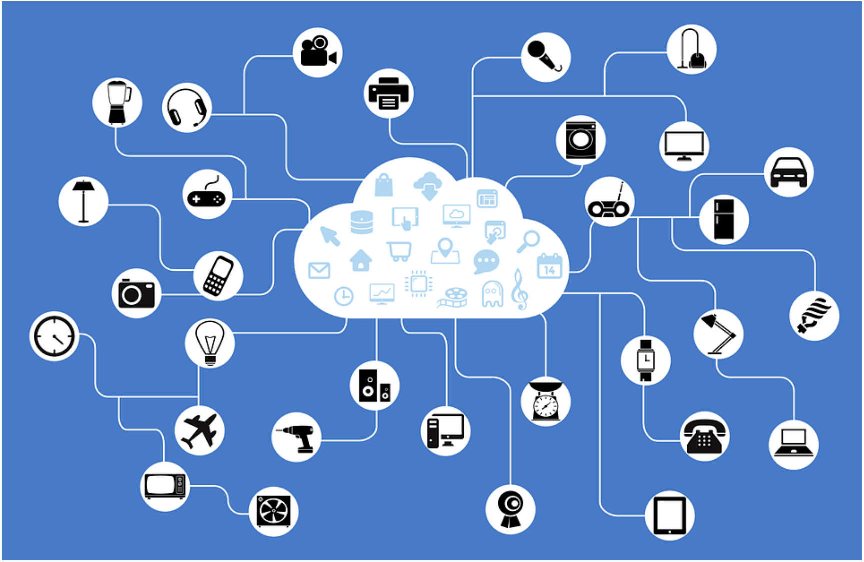
Towards Optimal Dynamic Graph Compression

Sebastian Krinninger

Universität Salzburg

Austrian Computer Science Day 2018

Graphs are Everywhere



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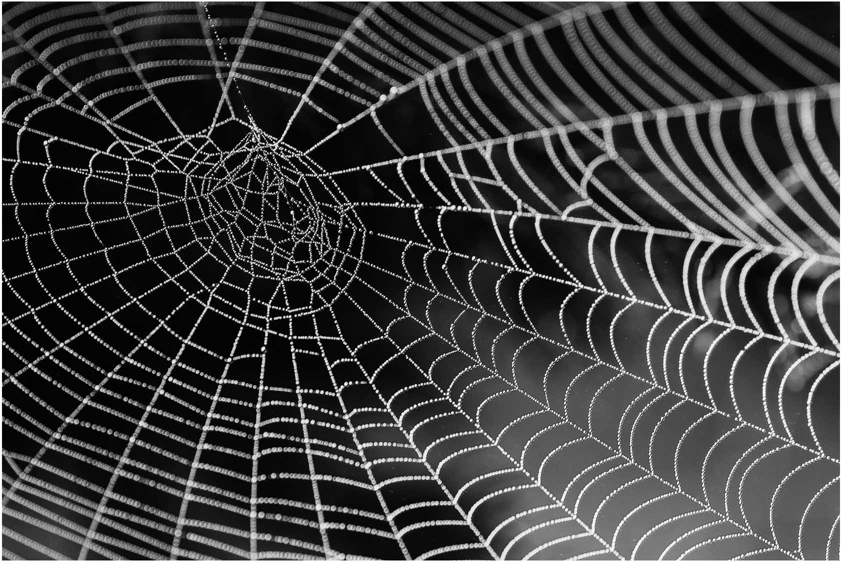
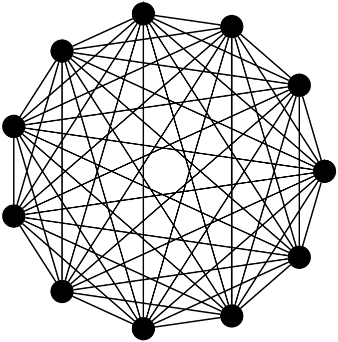


Graph Compression



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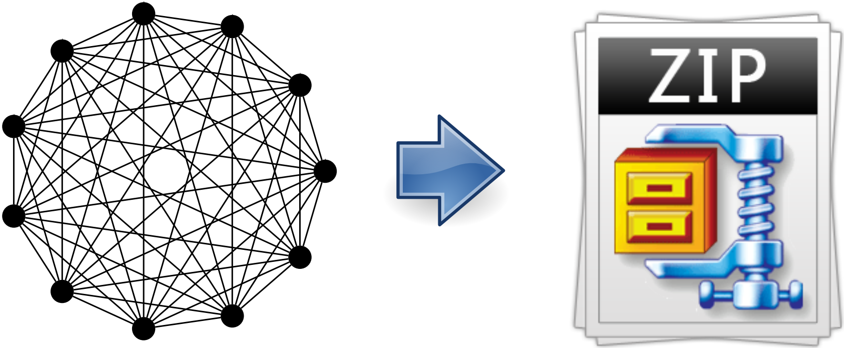

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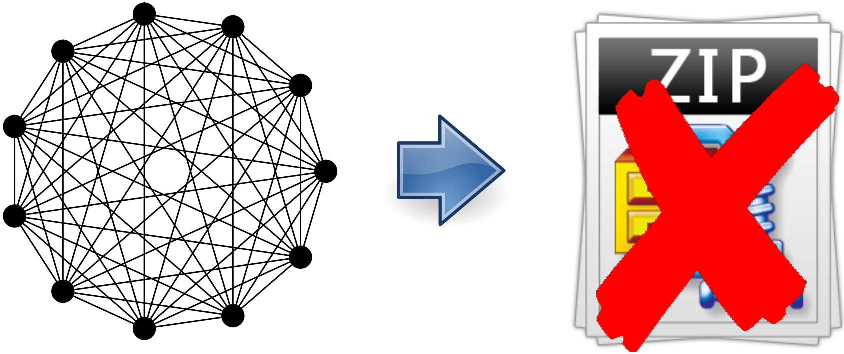

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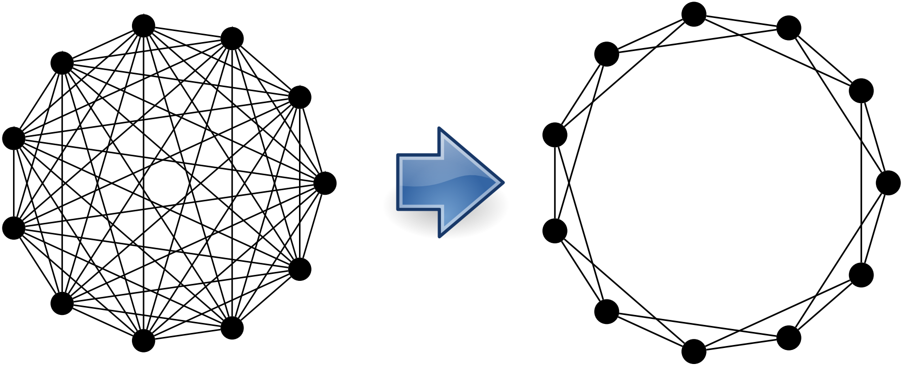

Goal: Semantic Compression

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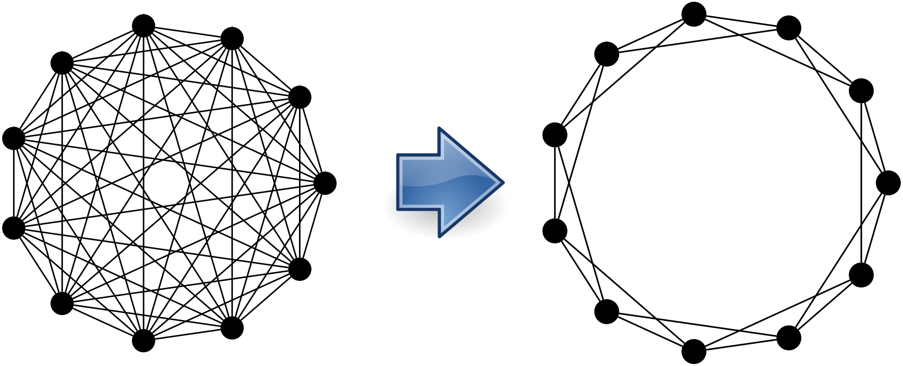

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Subgraph for algorithmic applications

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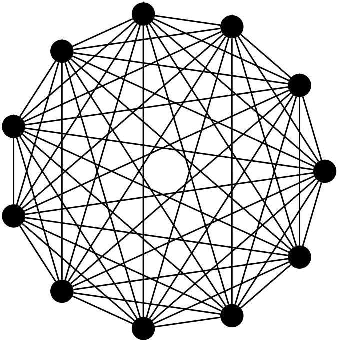
Thanks Christoph!

Lossy Compression



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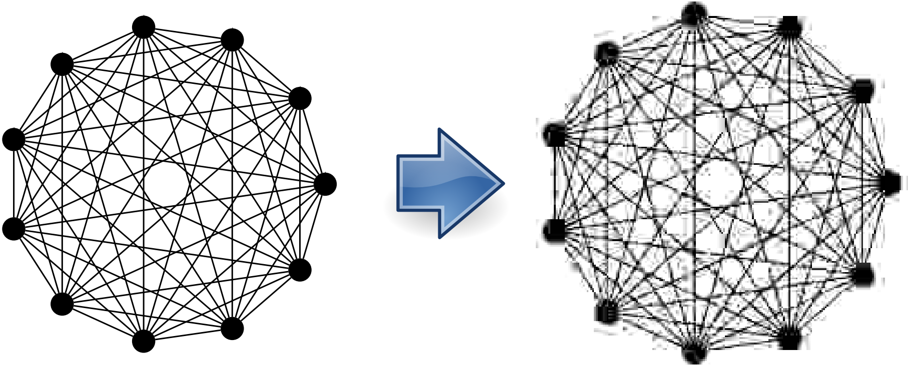

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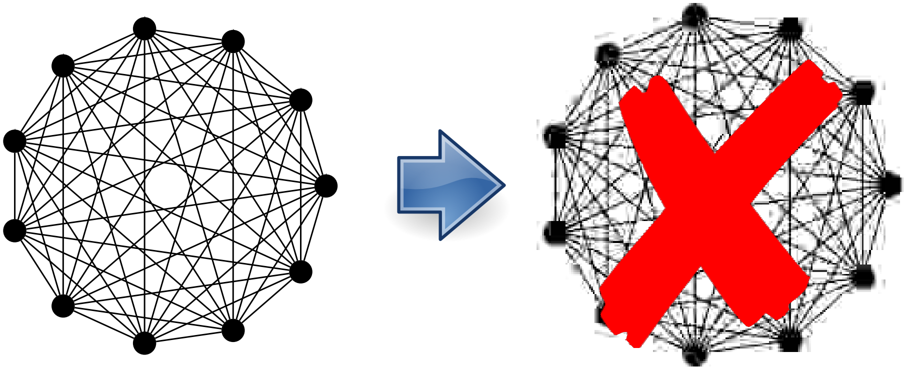

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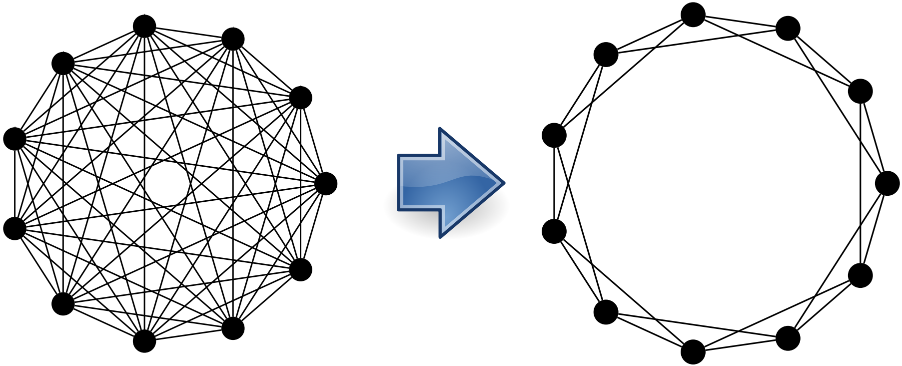

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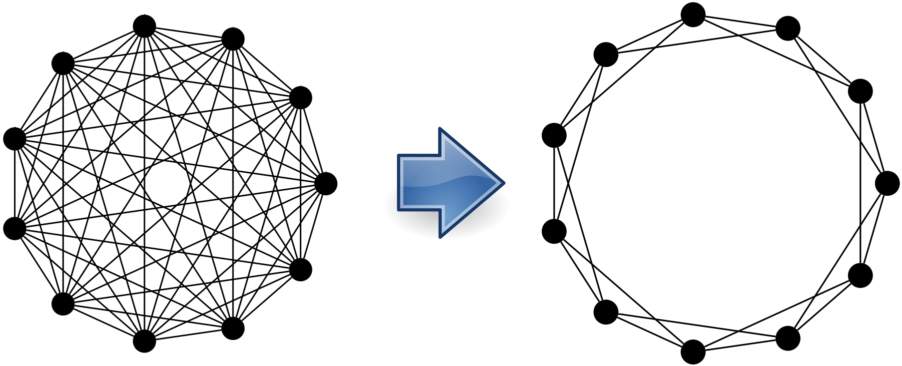

Cannot reconstruct original graph after compression
→ Compression at cost of approximation

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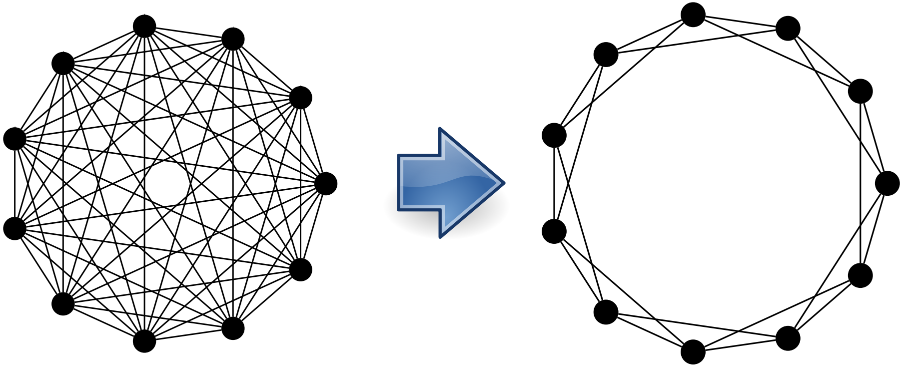

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When are two graphs approximately the same?
→ Problem-specific measures

Our World is not Static

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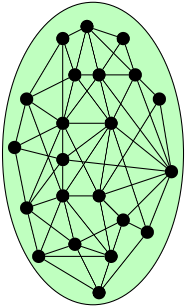
Goal: Fast recomputation of solution after each insertion/deletion of an edge

Dynamic Graph Compression

Input graph G



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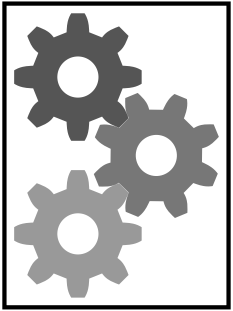

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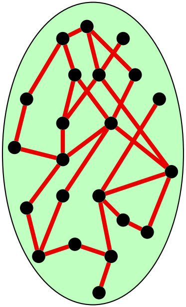

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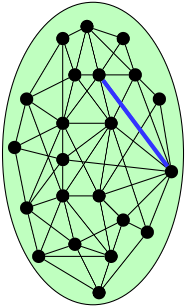

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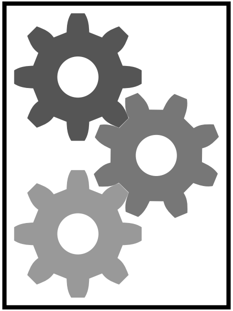

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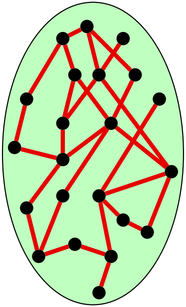

adversary inserts and deletes edges

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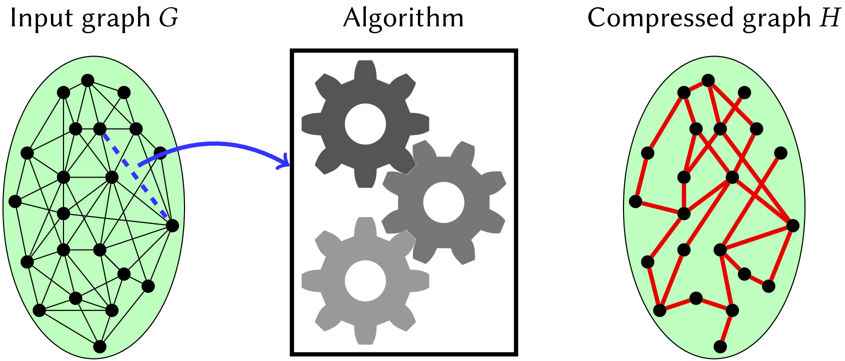

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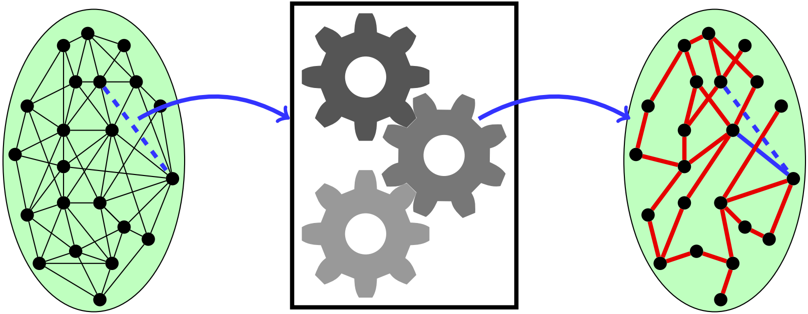

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Let's take a look under the hood!

Example 1: Distance-Preserving Compression

Definition

A *spanner* of *stretch* t of $G = (V, E)$ is a subgraph $H = (V, E')$ such that

$$dist_G(u, v) \le dist_H(u, v) \le t \cdot dist_G(u, v)$$

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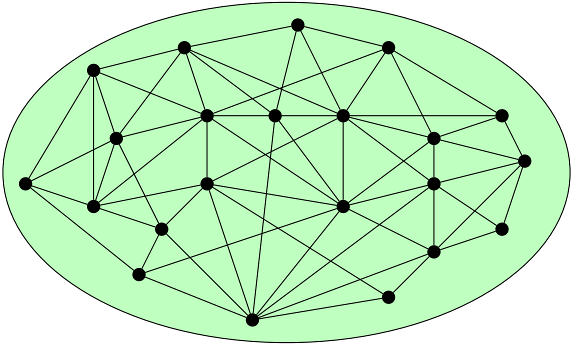

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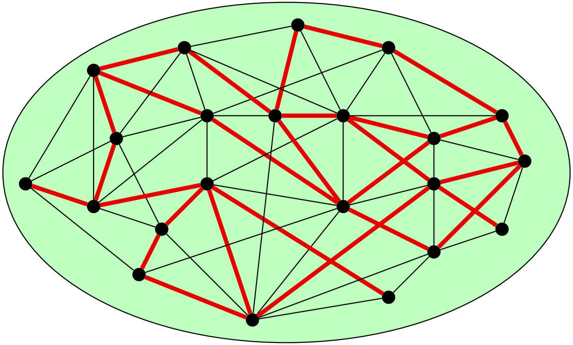

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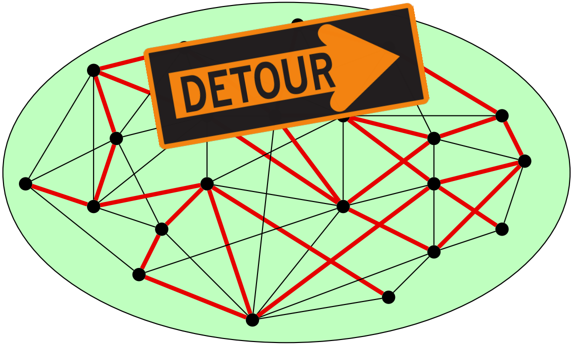

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In many applications: **boosting** approach for better approximation

Our Spanner Results

Theorem ([Baswana, Sarkar '08])

For every k, there is a dynamic algorithm that maintains a spanner of stretch $t = 2k - 1$

- *with $O(n^{1+1/k} k^8 \log^2 n)$ edges in amortized time $O(7^{k/2})$ per update,*
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For every k, there is a dynamic algorithm that maintains a $(2k-1)$-spanner with $O(n^{1+1/k} k \log^7 n \log\log n)$ edges in worst-case time $O(20^{k/2} \log^3 n)$ per update.

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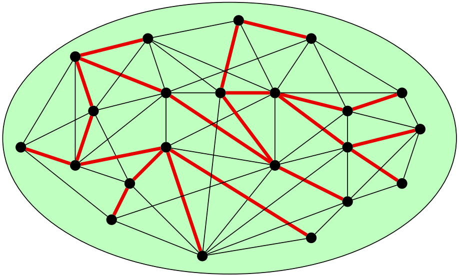

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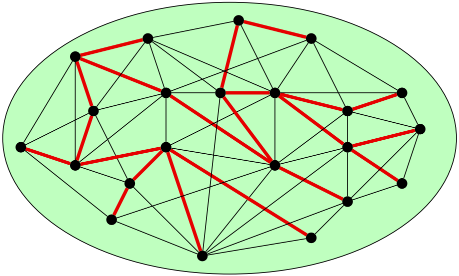

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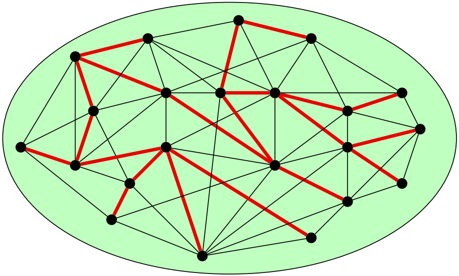

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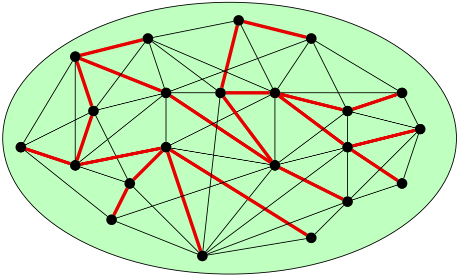

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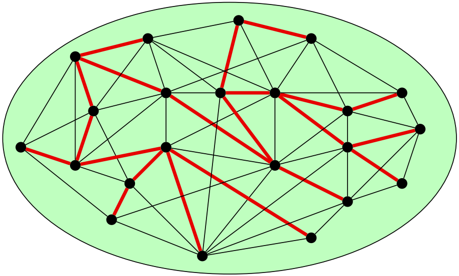

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Matches stretch of seminal static construction! [Alon/Karp/Peleg/West]

Example II: Cut-Preserving Compression

Definition ([Benczúr/Karger '00])

A $(1 \pm \epsilon)$-cut sparsifier of G is a weighted subgraph H such that, for every cut $(C, V \setminus C)$, the edges $E[C, V \setminus C]$ crossing the cut have weight

$$(1 - \epsilon) \cdot w_G(E[C, V \setminus C]) \leq w_H(E[C, V \setminus C]) \leq (1 + \epsilon) \cdot w_G(E[C, V \setminus C])$$

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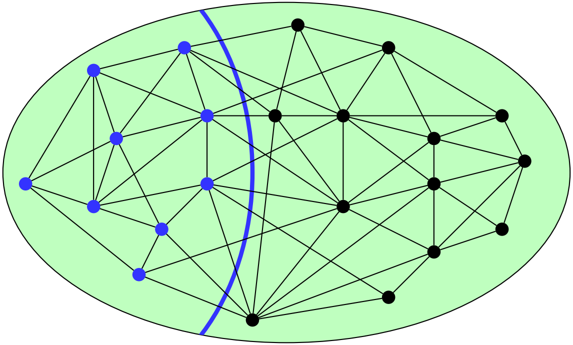

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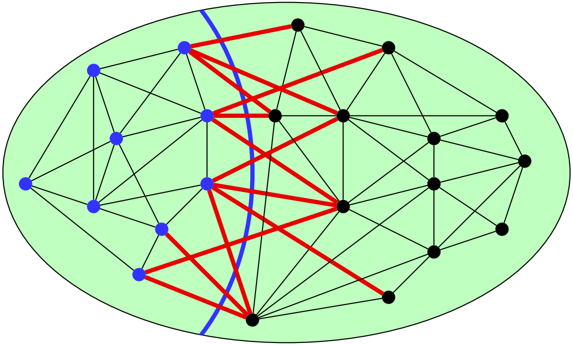

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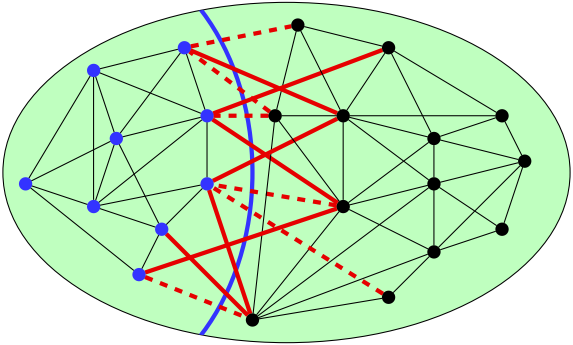

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Internally uses dynamic spanner with stretch $O(\log n)$

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Thank you!

Closing Words