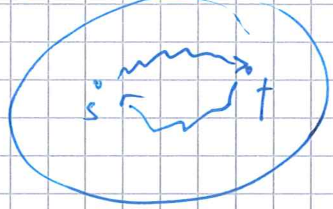
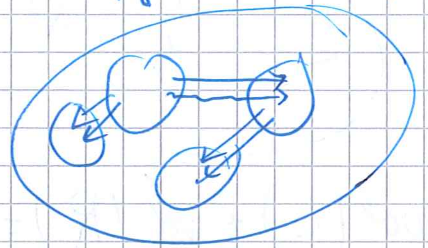
Computing and Testing Small Connectivity in Near-Linear Time and Queries via Fast Local Cut Algorithms

Definitions:

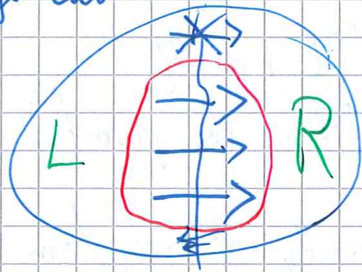
Strongly connected



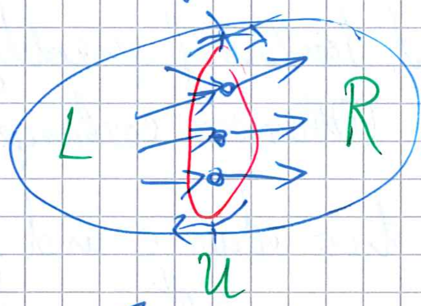
Recall: strongly connected components



Edge cut



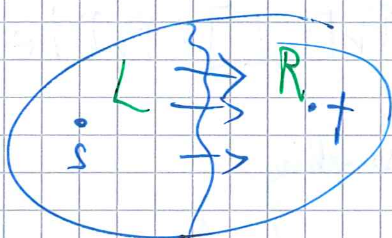
Vertex cut



size = #edges in cut

Induced partitions

s-t cuts

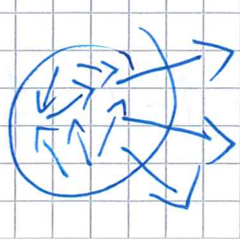


L and R are non-empty

If not $s \in L$ and $t \in R$, then s-t cut has size $\geq \lambda$
(cannot be $< \lambda$ by optimality of minimum cut)

Doubling approach: $\lambda \in [2^{i-1}, 2^i]$ for some $1 \leq i \leq \lceil \log n \rceil$
→ Algorithm with $k_i = 2^i$ will find min cut

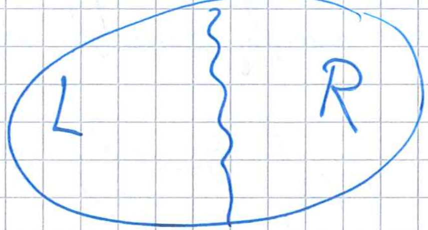
Volume



Case 1: min cut is balanced

$$vol(L) \geq \frac{m}{14k} \text{ and } vol(R) \geq \frac{m}{14k}$$

~~Without loss of generalit~~



$$vol(L) + vol(R) = m$$

→ One side must have volume $\geq \frac{m}{2}$

Without loss of generality: $vol(L) \geq \frac{m}{2}$
(Otherwise exchange L and R in the proof)

Sample two edges e and f uniformly at random

tail as s, tail t

$$Pr[s \in L] = \frac{vol(L)}{m} \geq \frac{1}{2} \qquad Pr[t \in R] = \frac{vol(R)}{m} \geq \frac{1}{6k}$$

$$Pr[s \in L \wedge t \in R] = Pr[s \in L] \cdot Pr[t \in R] \geq \frac{1}{2} \cdot \frac{1}{\cancel{6k}\,14k} = \frac{1}{28k}$$

independent sampling

s-t cut is min cut if $s \in L$ and $t \in R$
Algorithm is successful if this happens at least once during the $28k$ repetitions

$$Pr[\text{never "} s \in L \text{ and } t \in R \text{" during } 28k \text{ repetitions}]$$

$$\leq \left(1 - \frac{1}{28k}\right)^{28k} \leq \frac{1}{e} \leq \frac{1}{2}$$

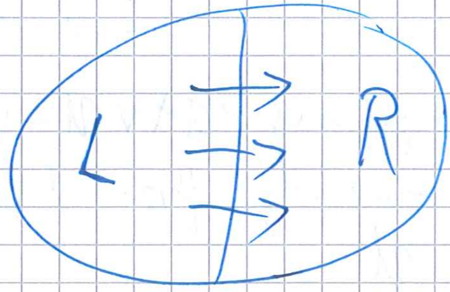
Bernoulli trial

$$Pr[\text{Algorithm successful}] \geq 1 - \frac{1}{2} = \frac{1}{2}$$

Case 2: min cut not balanced

$\mathrm{vol}(L) < \frac{m}{14k}$ or $\mathrm{vol}(R) < \cancel{\frac{m}{6k}}\ \frac{m}{14k}$

Consider min-cut



If $\mathrm{vol}(L) < \frac{m}{6k}$

$\cancel{\Delta_i \leq \mathrm{vol}(L) \leq \Delta_{i+1}}$

There is some i s.t. $2^{i-1} \leq \mathrm{vol}(L) \leq \underbrace{2^{i} - 1}_{=\Delta_i}$
(in particular $1 \leq i \leq r$)

Sample edge e, let s be the tail

$\Pr[s \in L] = \frac{\mathrm{vol}(L)}{m} \geq \frac{2^{i-1}}{m}$

Probability that $s \in L$ for at least one of the $\lceil \frac{m}{\Delta_i} \rceil$ repeats:

$\geq 1 - \frac{1}{e} \geq 0.6$ (analysis like before)

If $s \in L$, then s contained in λ-out component of volume $\leq \Delta_i$ with probability $\geq \frac{5}{6}$

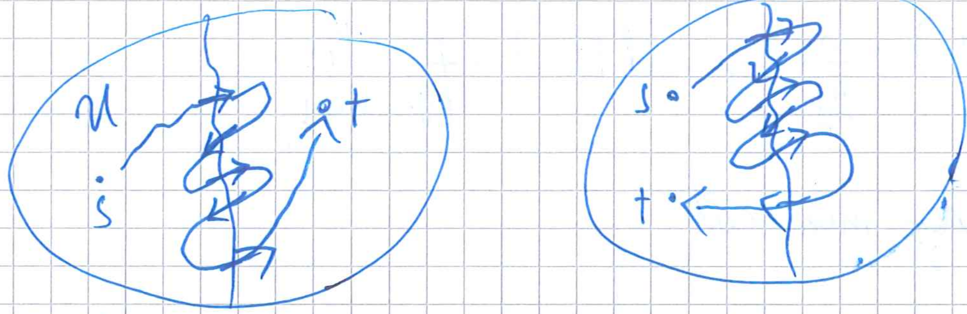
$\Rightarrow$ local procedure finds λ-out component U of volume $\leq \cancel{8}\,7k\Delta_i < m$ as $\Delta_i \leq 2^{r} - 1 \leq 2^{\log \frac{m}{7k}} - 1 < \frac{m}{7k}$

$\Rightarrow U \neq V$ and there is a cut of size λ that disconnects U from V

$\Rightarrow$ Overall success probability $\geq 0.6 \cdot \frac{5}{6} = \frac{1}{2}$

If $\mathrm{vol}(R) < \frac{m}{6k}$: same with in-component of R

Claim 1



Path from s to t might cross ~~...~~ U – $V \setminus U$ several times, but one more edge from U to $V \setminus U$ than from $V \setminus U$ to U

$\rightarrow$ after reversal # edges from U to $V \setminus U$ decreases by 1

Claim 2:

If DFS visits less than specified # vertices, then DFS visits all vertices currently reachable from s

~~Let~~ Let U = # vertices reachable from s = vertices visited in DFS

No edges from U to ~~...~~ $V \setminus U$ in current graph

- By Claim 1 this # edges has reduced by ≤ 1 in each previous ~~all~~ iteration
- Previously l iterations

$\Rightarrow$ Initially $\leq l$ edges from U to $V \setminus U$

$\rightarrow$ U is l-out component

Claim 3: By Claim 1, algorithm succeeds if always $t \in V \setminus U$

- ~~Let F be set of ... edges processed in~~ in first l iterations
- Let U be l out component (in iteration $l+1$: 0 edges from U to $V \setminus U$)
- ~~vol(...)~~ $vol_G(U) \leq \Delta$ (in ~~original~~ initial graph)
- Let F be set of edges processed by DFS, $vol(F) = 6k\Delta$
- $Pr[t \in U] \leq \frac{vol(U)}{vol(F)} \leq \frac{\Delta}{vol(F)} = \frac{\Delta}{6k\Delta} = \frac{1}{6k}$
- $Pr[t \in U \text{ in one of the first } l \text{ iterations}] \leq l \cdot \frac{1}{6k} \leq k \cdot \frac{1}{6k} = \frac{1}{6}$ (Union Bound)