Faster Cut Sparsification of Weighted Graphs

Joint work with Sebastian Forster

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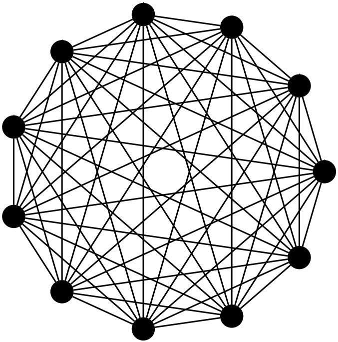
Sparsification



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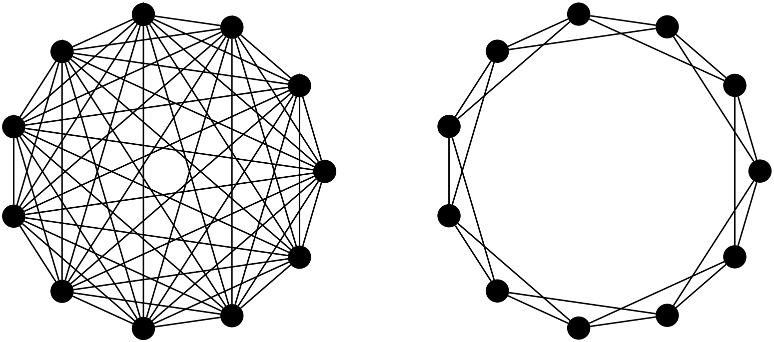
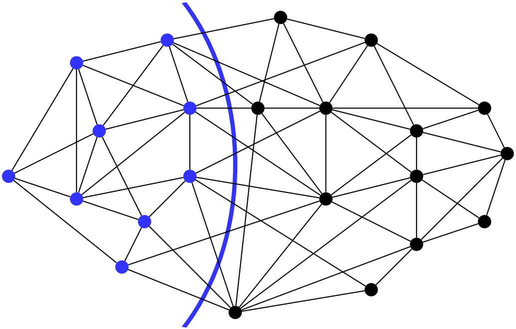


Cuts and Cut Sparsification[1]



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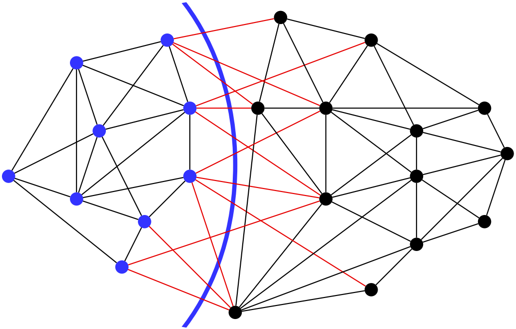

Weight of cut: $w_G(C)$

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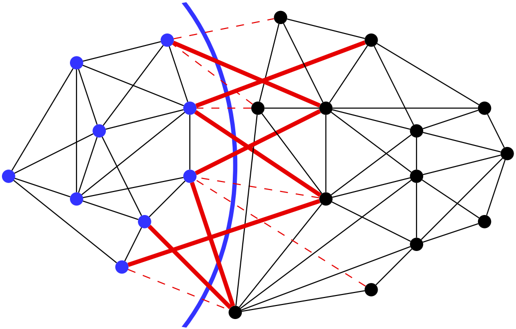

Weight of cut: $w_G(C)$
Weight of sparsified cut $w_H(C)$

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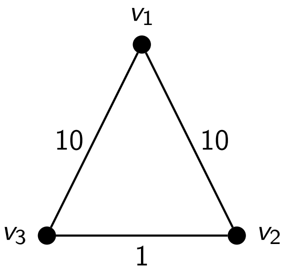

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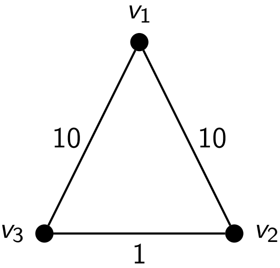

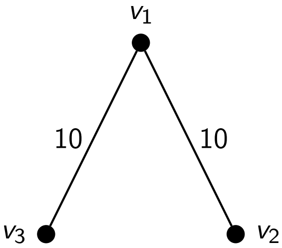

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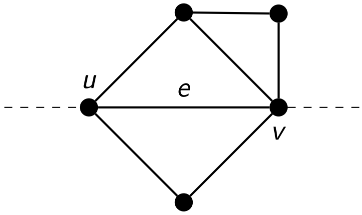

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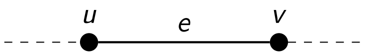

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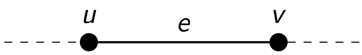

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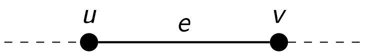

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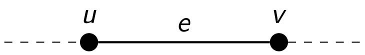

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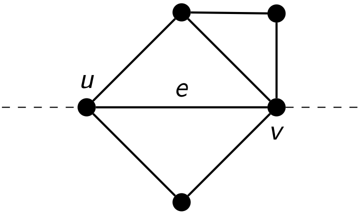

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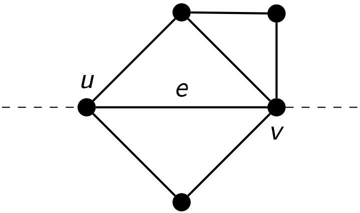

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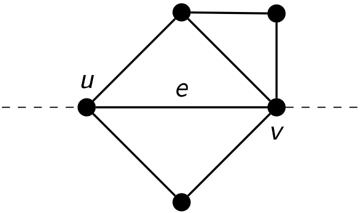

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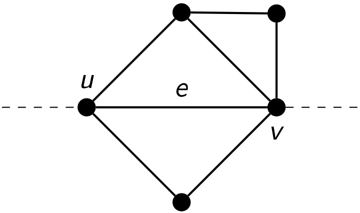

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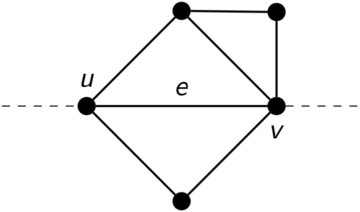

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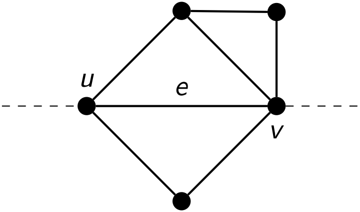

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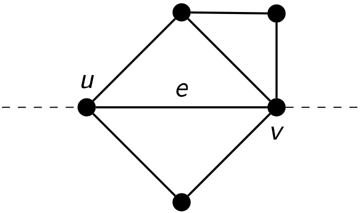

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Connectivity Estimators λ_e

Sample with $p_e \sim \frac{c\gamma \cdot \log n}{\lambda_e \epsilon^2}$, for some $\lambda_e \leq c_e$

[FHHP11] Edge Connectivity

For graphs with polynomially bounded integer weights.

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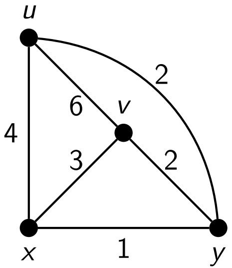

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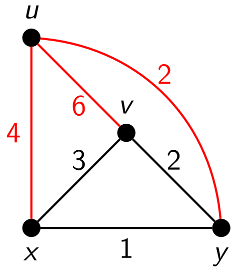

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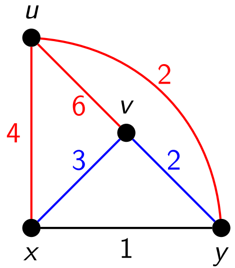

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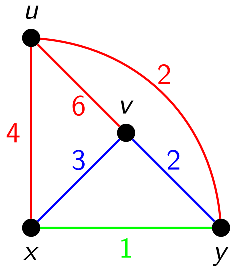

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MSF Indices and Connectivity

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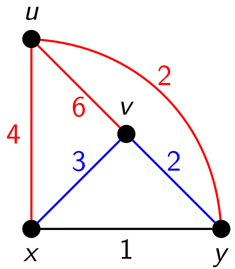

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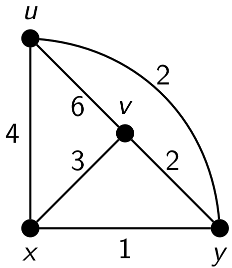

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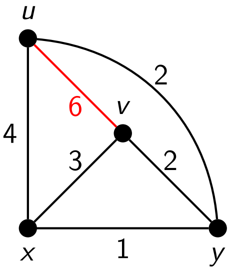

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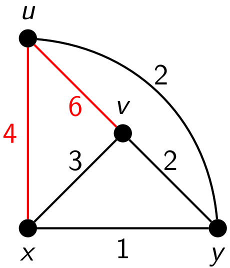

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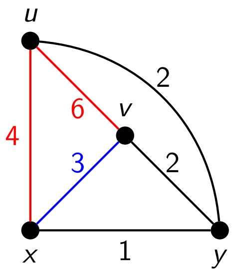

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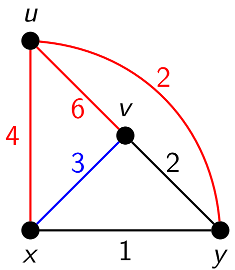

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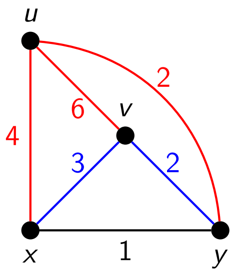

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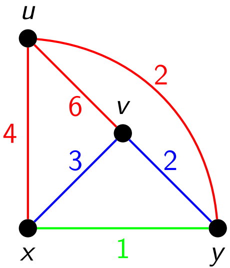

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     - Pay $\alpha(n)$ for maintaining Union-Find data structures

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1. Compute MSF indices upto $M = n$ in time $O(m\alpha(n)\log n)$.
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Goal is time $O(m\alpha(n)\log(m/n))$ and size $O(n \log n/\epsilon^2)$.

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Conclusion

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$G = (V, E)$ weighted, $\epsilon > 0$. There exists an algorithm that computes a $(1 \pm \epsilon)$-cut sparsifier for G with high probability, in time $O(m\alpha(n) \log(m/n))$ and with size is $O\left(n \log n/\epsilon^2\right)$.

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