Exercise 1 (10 points) Let $A$ and $B$ be two $n \times n$ matrices with integer entries in $\{-M, -M+1, \ldots, M - 1, M\}$. Show that the min-plus matrix product of $A$ and $B$ can be computed in time $O(M^2 \cdot n^\omega)$. Here $\omega$ is the exponent of matrix multiplication.

Remark: This problem can actually be solved in time $(M \cdot n^\omega \cdot \text{poly}(\log M, \log n))$.

Exercise 2 (10 points) In the $k$-Clique problem, we are given an unweighted undirected graph $G$ and are asked to decide if $G$ contains a $k$-clique (i.e., a set of $k$ vertices which are pairwise adjacent). Show that if $k$ is divisible by 3, then $k$-Clique can be solved in time $O(n^\omega k^3)$.

Hint: Reduce the problem to detecting a triangle in a graph with $O(n^k)$ vertices.

Remark: This is the best running time known for this problem.

Exercise 3 (10 points)

In the Zero-Weight 3-Star problem, we are given a weighted 4-partite graph $G = (V_1 \cup V_2 \cup V_3 \cup V_4, E)$, where $|V_1| = |V_2| = |V_3| = |V_4| = n$, and are asked to decide whether there are $v_1 \in V_1$, $v_2 \in V_2$, $v_3 \in V_3$, and $v_4 \in V_4$ such that $w(v_1, v_2) + w(v_1, v_3) + w(v_1, v_4) = 0$.

Show that Zero-Weight 3-Star has an algorithm with running time $O(n^{3-\epsilon})$ (for some $\epsilon > 0$) if and only if there is an algorithm with running time $O(n^{3-\delta})$ (for some $\delta > 0$) that decides if at least one of $n$ given 3SUM instances has a solution.

Exercise 4 (10 points)

In the Segment Visibility problem, we are given a set $S$ of $n$ line segments in the plane and two distinguished line segments $a$ and $b$, and are asked to decide whether there are points $p$ on $a$ and $q$ on $b$ such that the line through the points $p$ and $q$ does not intersect any line segment in $S$ (i.e., we want to check whether $a$ is “visible” from $b$).

Show that if Segment Visibility can be solved in time $O(n^{2-\epsilon})$ for some $\epsilon > 0$, then 3SUM can be solved in time $O(n^{2-\delta})$ for some $\delta > 0$. 

Prove all your claims!