Exercise 1 (10 points)
The Metricity problem is defined as follows: Given an $n \times n$ matrix $A$ with entries in $\{0, \ldots, \lfloor n^c \rfloor\}$ for some constant $c > 0$, decide whether $A[i, j] \leq A[i, k] + A[k, j]$ for all $i, j, k \in \{1, \ldots, n\}$.
Prove that Metricity is equivalent to APSP under subcubic reductions.
*Hint*: Reduce Metricity to Min-Plus Product and reduce Negative Triangle to Metricity.

Exercise 2 (10 points)
Consider a directed graph $G = (V, E, w)$ with positive integer weights in the range $w(e) \in \{1, 2, \ldots, W\}$ for each edge $e \in E$. The Betweenness Centrality of a node $v \in V$ is the number of pairs $s, t$ such that $v$ lies on a shortest path from $s$ to $t$, i.e.,

$$BC_G(v) = |\{(s, t) : s, t \in V \setminus \{v\}, s \neq t, \text{dist}(s, t) = \text{dist}_G(s, v) + \text{dist}_G(v, t)\}|.$$ 

and the Diameter of $G$ is the maximum distance between any pair of nodes, i.e.,

$$\text{diam}(G) = \max_{s, t \in V} \text{dist}_G(s, t).$$

Show that if there is an algorithm for computing the Betweenness Centrality of a node in a graph with positive edge weights running in time $T(n, m)$, then there is an algorithm for computing the diameter of a graph with positive edge weights running in time

$$O(\bar{T}(O(n), O(m + n)) \log(nW) + m).$$

*Hint*: Introduce a dummy node and perform binary search to find the value of the diameter.

Exercise 3 (10 points)
Give an algorithm for Orthogonal Vectors with running time $O(n^{\omega})$, i.e., an algorithm that (theoretically) outperforms the naive $O(n^2 d)$-time algorithm in the high-dimensional regime.

Exercise 4 (10 points)
Work out the subcubic reduction from All-Pairs Triangle Detection to Triangle Detection mentioned in class. Note: As a consequence, this will prove that if there is a subquadratic combinatorial algorithm for Triangle Detection, then there also is a subquadratic combinatorial algorithm for All-Pairs Triangle Detection.