PS Geometric Modeling
Homework Assignment Sheet I (Due 16-Oct-2020)

Assignment 1 Prove: If x_1, x_2 are the real roots of the polynomial $x^2 + px + q$, with $p, q \in \mathbb{R}$, then

$$x_1 + x_2 = -p \qquad \text{and} \qquad x_1 \cdot x_2 = q.$$

How can this fact be used in practice?

Assignment 2 Let $a, b, c, x_1, x_2 \in \mathbb{R}$, with $a \neq 0$ and suppose that the polynomial equation $ax^2+bx+c = 0$ has real solutions. Prove the following claim: The polynomial ax^2+bx+c can be factorized into $a(x-x_1)(x-x_2)$ if and only if x_1, x_2 are the (real) solutions of $ax^2+bx+c = 0$.

Assignment 3 Let $p, q, x_1, x_2 \in \mathbb{R}$ such that

$$x_1 + x_2 = -p \qquad \text{and} \qquad x_1 \cdot x_2 = q.$$

Prove that the polynomial equation $x^2 + px + q = 0$ has x_1, x_2 as its real solutions.

Assignment 4 Rewrite the monomials of the polynomial

$$5+3x^2+xyz+x^2yz+x^3yz+y^2z+xy^2z+xy^4z^2+x^2y^2z^2+x^4y^2z^2+z^3+x^3z^3+2xyz^3+x^2yz^3$$

such that it becomes apparent that the polynomial belongs to $((\mathbb{R}[x])[y])\,[z]$. What is the degree of the polynomial?

New homework assignments can be retrieved from `www.cosy.sbg.ac.at/~held/teaching/geom_mod/ps_bsp.pdf`, and should be solved prior to the due dates. You are not required to hand in your assignments, unless explicitly requested for some particular assignment.

PS Geometric Modeling
Homework Assignment Sheet II (Due 23-Oct-2020)

Assignment 5 Specify functions $f, g, h, i : I \to \mathbb{R}$, for some open interval $I \subset \mathbb{R}$ such that

1. f is continuous on I but cannot be differentiated on all of I,
2. g can be differentiated everywhere on I but g' is not continuous on I,
3. h can be differentiated once continuously on I, but cannot be differentiated twice on all of I,
4. i is smooth on I.

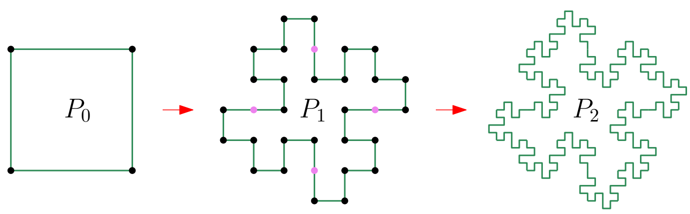
(If it helps then you are welcome to use L'Hôpital's rule.)

Assignment 6 Compute $\frac{\partial^2 f}{\partial x \partial y}(x, y)$ and $\frac{\partial^2 f}{\partial y \partial x}(x, y)$ and check that they are identical for all $(x, y) \in \mathbb{R} \times \mathbb{R}^+$, for

$$f(x, y) := xe^{2y} + \sqrt{e^y \sin^2(\sqrt{y + \log y})} + \sqrt{2 + y^2 \cos^2 y}.$$

Assignment 7 Let $S \subseteq \mathbb{R}$ be an open set. Recall that a function $f : S \to \mathbb{R}^n$ is continuous at an interior point $x_0 \in S$ if $\lim_{h \to 0} f(x_0 + h) = f(x_0)$. Prove that f is continuous at x_0 if it is differentiable at x_0.

Assignment 8 Let P_0 be a square of edge length 1. Now suppose that we generate a closed polygon P_1 by replacing each edge of P_0 by eight shorter edges, as shown below: All new edges are of the same length, and except for four pairs of new edges (which are collinear) all new edges meet at right angles. Inductively we generate P_{n+1} from P_n by replacing each edge of P_n by appropriate copies of eight new edges.



What is the circumference $C(n)$ and the area $A(n)$ of P_n? What are the dimensions of the smallest axis-aligned bounding box of the union $\cup_{n \in \mathbb{N}} P_n$ of all such polygons?

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PS Geometric Modeling
Homework Assignment Sheet III (Due 30-Oct-2020)

Assignment 9 Consider the curve $\gamma\colon \mathbb{R} \to \mathbb{R}^2$ with $\gamma(t) := (t^2, t^3)$. (1) Sketch the (image of the) curve for $t \in [-2, 2]$. (2) Is γ smooth? (3) Is γ regular? (4) What is the maximum $k \in \mathbb{N}$ such that the restriction of γ to $\mathbb{R}^+$ is regular of order k? (5) Compute the arc length of the restriction of γ to $[0, 1]$.

Assignment 10 Consider the curves $\beta\colon [0, 1] \to \mathbb{R}^2$ and $\gamma\colon [0, 2] \to \mathbb{R}^2$ with

$$\beta(t) := \begin{pmatrix} \cos \frac{\pi}{2} t \\ \sin \frac{\pi}{2} t \end{pmatrix} \qquad \text{and} \qquad \gamma(t) := \begin{pmatrix} \cos\left(\frac{3\pi}{2} - \frac{\pi}{4} t\right) \\ 2 + \sin\left(\frac{3\pi}{2} - \frac{\pi}{4} t\right) \end{pmatrix}.$$

Show that β and γ are G^1-continuous but not C^1-continuous at the joint $(0, 1)$. Furthermore, find a re-parametrization α of β such that α and γ are C^1-continuous at the joint. Make sure to check that α and γ are indeed equivalent!

Assignment 11 Consider two intervals $I, J \subseteq \mathbb{R}$ and a regular C^1 curve $\gamma\colon I \to \mathbb{R}^3$. We denote the coordinate functions of γ by $\gamma_x, \gamma_y, \gamma_z$ and assume that $\gamma_y(s) = 0$ for all $s \in I$. Let $\alpha\colon I \times J \to \mathbb{R}^3$ with $J := [0, 2\pi]$ and

$$\alpha(s, t) := (\gamma_x(s) \cdot \cos t, \gamma_x(s) \cdot \sin t, \gamma_z(s)).$$

Under which conditions on γ is α regular? Which surface do we get for $I := [-\pi/2, \pi/2]$ and $\gamma_x(s) := \cos s$, $\gamma_z(s) := \sin s$?

Assignment 12 Determine the equation of a plane that is tangential to the graph (surface) of $f\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, with $f(x, y) := e^{-x^2-y^2}$, for $x := 1$ and $y := -1$.

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PS Geometric Modeling
Homework Assignment Sheet IV (Due 06-Nov-2020)

Assignment 13 Let $n \in \mathbb{N}$ and $p_0, \ldots, p_n \in \mathbb{R}^2$ be arbitrary but fixed. Prove explicity that

$$\sum_{i=0}^{n} B_{i,n}(t)p_i = \sum_{i=0}^{n} B_{i,n}(1-t)p_{n-i}.$$

for all $t \in [0, 1]$.

Assignment 14 Consider a cubic Bézier curve with control points p_0, p_1, p_2, p_3 and denote the control polygon (p_0, p_1, p_2, p_3) by P. Assume that there exists a point q on P such that P has a point symmetry relative to q. What can we say about the the intersection of the Bézier curve and P?

Assignment 15 Consider a Bézier curve of degree $n \in \mathbb{N}$ and assume that all $n+1$ control points $p_0, p_1, \ldots, p_n$ are collinear. Prove explicitly (without resorting to the convex hull property (Lemma 91)) that all points of the Bézier curve lie on the supporting line of $p_0, p_1, \ldots, p_n$.

Assignment 16 Consider the degree-two Bézier curve $\mathcal{B}_1$ defined by the control points

$$\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}2\\1\end{pmatrix}$$

and the degree-four Bézier curve $\mathcal{B}_2$ defined by the control points

$$\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}2\\1\end{pmatrix}, \begin{pmatrix}3\\2\end{pmatrix}, \begin{pmatrix}4\\2\end{pmatrix}.$$

Someone conjectures that $\mathcal{B}_2|_{[0,1/2]}$ would equal $\mathcal{B}_1$. Prove or disprove this conjecture. (A mere plot alone is not good enough!)

PS Geometric Modeling
Homework Assignment Sheet V (Due 13-Nov-2020)

Assignment 17 Consider the setting of Assignment 15. What may happen to the Bézier curve if the control points do not appear in sorted order along their supporting line? Why does this not contradict the fact that Bézier curves are smooth?

Assignment 18 Prove explicitly (by resorting to the requirements stated in Theorem 95 ("polar form") that the four functions listed in the table below Lemma 96 form polar forms for the case $n = 3$.

Assignment 19 Consider a Bézier curve $\mathcal{B}$ of degree $n \in \mathbb{N}$ with control points $p_0, p_1, \ldots, p_n$, and use de Casteljau's algorithm to subdivide it into two Bézier curves $\mathcal{B}_1, \mathcal{B}_2$ at a point $\mathcal{B}(t_0)$, for some $t_0 \in [0.1]$. Under which conditions are $\mathcal{B}_1, \mathcal{B}_2$ C^1 continuous at $\mathcal{B}(t_0)$?

Assignment 20 Consider the Bézier surface

$$\mathcal{S}(u, v) := \sum_{i=0}^{n} \sum_{j=0}^{m} B_{i,n}(u) B_{j,m}(v) p_{i,j} \qquad \text{for } (u, v) \in [0, 1] \times [0, 1],$$

with the control points $p_{i,j}$ for $0 \le i \le n$ and $0 \le j \le m$. Resort to the definition of a tangent plane (Def. 74) to verify that the tangent plane in $\mathcal{S}(1, 1)$ is spanned by the vectors $p_{n,m} - p_{n-1,m}$ and $p_{n,m} - p_{n,m-1}$.

PS Geometric Modeling
Homework Assignment Sheet VI (Due 20-Nov-2020)

Assignment 21 Compute all B-spline basis functions of degrees zero and one that are defined by the knot vector $\tau := (0, 0, 0, 0.3, 0.5, 0.5, 0.6, 1, 1, 1)$.

Assignment 22 Let τ be a bi-infinite uniform knot vector. Prove that $N_{i,2,\tau}$ is continuous for all $i \in \mathbb{Z}$.

Assignment 23 Let $\tau := (t_i)_{i\in\mathbb{Z}}$ be a bi-infinite uniform knot vector with $t_i := i$ for all $i \in \mathbb{Z}$. Prove that

$$N'_{i,1}(t) = N_{i,0}(t) - N_{i+1,0}(t)$$

for all $i \in \mathbb{Z}$. In order to simplify matters you are welcome to ignore discontinuity issues (at the knots) as much as this is possible.

Assignment 24 Let $\tau := (t_i)_{i\in\mathbb{Z}}$ be a bi-infinite uniform knot vector with $t_i := i$ for all $i \in \mathbb{Z}$. Use induction (on the degree k) and the recursive definition of the B-spline basis functions to prove that

$$N'_{i,k}(t) = \frac{k}{t_{i+k} - t_i} N_{i,k-1}(t) - \frac{k}{t_{i+k+1} - t_{i+1}} N_{i+1,k-1}(t)$$

for all $k \in \mathbb{N}$ and all $i \in \mathbb{Z}$. In order to simplify matters you are welcome to ignore discontinuity issues (at the knots) as much as this is possible.

PS Geometric Modeling
Homework Assignment Sheet VII (Due 27-Nov-2020)

Assignment 25 Let $\mathcal{P}$ be a B-spline curve of degree one defined by $n+1$ control points $p_0, p_1, \ldots, p_n$, over the clamped knot vector $\tau := (t_0, t_1, \ldots, t_{n+2})$, where all knots except t_0, t_1, t_n, t_{n+1} are distinct. In the lecture I claimed that $\mathcal{P}$ equals the control polygon $(p_0, p_1, \ldots, p_n)$. Prove explicitly (without resorting to a convex-hull property, such as Lem. 136) that, for all $i \in \{1, 2, \ldots, n-1, n\}$, $\mathcal{P}|_{[t_i, t_{i+1}[}$ equals $\overline{p_{i-1}p_i}$ (excluding p_i).

Assignment 26 Consider the clamped degree-two B-spline curve $\mathcal{P}$ defined by the control points

$$\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}3\\2\end{pmatrix}, \begin{pmatrix}5\\1\end{pmatrix}, \begin{pmatrix}5\\0\end{pmatrix}$$

and the knot vector $\tau := (0, 0, 0, 1/4, 1/2, 3/4, 1, 1, 1)$. Does the line segment

$$\left\{ \begin{pmatrix}1\\1\end{pmatrix} + \lambda \begin{pmatrix}2\\0\end{pmatrix} : \lambda \in [0, 1] \right\}$$

intersect $\mathcal{P}$? Would we get different results for other clamped (possibly non-uniform) knot vectors instead of τ?

Assignment 27 Let $\mathcal{P}$ be a B-spline curve of degree k defined by $n+1$ control points $p_0, p_1, \ldots, p_n$ and the knot vector $\tau := (t_0, t_1, \ldots, t_{n+k+1})$, for $n \in \mathbb{N}$ and $k \in \mathbb{N}$. Suppose that all knots are distinct (except for possibly the first and last $k+1$ knots if $\mathcal{P}$ is clamped). Lemma 122 implies that $\mathcal{P}$ is a C^{k-1} curve. Why is C^{k-1} continuity not contradicted by the fact that $\mathcal{P}$ contains a "corner" at p_i, for $i \in \mathbb{N}$ with $k \leq i < n$, if $p_{i-k+1} = p_{i-k+2} = \ldots = p_i$? Let $k := 2$ and argue that C^1 continuity is preserved at p_i even if $p_{i-1} = p_i$.

Assignment 28 Let $\mathcal{P}$ be a B-spline curve of degree two defined by $n+1$ control points with position vectors $p_0, p_1, \ldots, p_n$, over the clamped knot vector $\tau := (t_0, t_1, \ldots, t_{n+3})$. Prove that, for all $i \in \{3, 4, \ldots, n-1, n\}$, the tangent to $\mathcal{P}$ at $\mathcal{P}(t_i)$ is parallel to the leg $\overline{p_{i-2}, p_{i-1}}$.

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PS Geometric Modeling
Homework Assignment Sheet VIII (Due 04-Dec-2020)

Assignment 29 Let $\mathcal{P}$ be a B-spline curve of degree 2 defined by $n+1$ control points with position vectors $p_0, p_1, \ldots, p_n$, over the clamped uniform knot vector $\tau := (t_0, t_1, \ldots, t_{n+3})$. Prove that, for all $i \in \{3, 4, \ldots, n-1, n\}$, the tangent to $\mathcal{P}$ at $\mathcal{P}(t_i)$ is given by the supporting line of the leg $\overline{p_{i-2}, p_{i-1}}$.

Assignment 30 Consider a clamped uniform B-spline $\mathcal{P}$ of degree one for the thirteen control points

$$\left\{ \begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}1\\3\end{pmatrix}, \begin{pmatrix}3\\5\end{pmatrix}, \begin{pmatrix}5\\5\end{pmatrix}, \begin{pmatrix}6\\4\end{pmatrix}, \begin{pmatrix}5\\2\end{pmatrix}, \begin{pmatrix}3\\2\end{pmatrix}, \begin{pmatrix}3\\1\end{pmatrix}, \begin{pmatrix}11\\1\end{pmatrix}, \begin{pmatrix}8\\3\end{pmatrix}, \begin{pmatrix}8\\5\end{pmatrix}, \begin{pmatrix}10\\6\end{pmatrix}, \begin{pmatrix}4\\7\end{pmatrix} \right\}$$

and the knot vector

$$\tau := (-2, -2, -1, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 10).$$

We insert $^{11}/_{2}$ into τ and obtain a new knot vector $\tau^\star$. Determine new control points $p_i^\star$ such that the B-spline curve $\mathcal{P}^\star$ relative to $\tau^\star$ and $p_i^\star$ equals $\mathcal{P}$.

Assignment 31 Prove that each $p_i^\star$ defined in Böhm's Lem. 146 lies either at a vertex or within the (relative) interior of an edge of the original control polygon.

Assignment 32 Let $\mathcal{P}$ be a B-spline curve of degree 2 defined by $n+1$ control points with position vectors $p_0, p_1, \ldots, p_n$ and the uniform knot vector $\tau := (t_0, t_1, \ldots, t_{n+3})$, for $n \in \mathbb{N}\backslash\{1\}$. Prove explicitly that $\mathcal{P}(t_k) = \lim_{t \nearrow t_{n+1}} \mathcal{P}(t)$ and that $\mathcal{P}$ is C^1 at this joining point if $p_0 = p_{n-1}$ and $p_1 = p_n$. (You are welcome to ignore the formal problems regarding t_{n+1} and just evaluate all necessary terms for t_{n+1} "in the natural way", thereby disregarding any limits.)

PS Geometric Modeling
Homework Assignment Sheet IX (Due 11-Dec-2020)

Assignment 33 In $\mathbb{R}^2$ we consider the hyperbola $x \cdot y = 1$. Compute the points of intersection of this hyperbola with the lines $y = 1$ and $y = 0$, using (a) inhomogenous coordinates, and (b) homogeneous coordinates.

Assignment 34 The lecture slides claim that

$$\left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}\right) \qquad \text{for } t \in \mathbb{R}$$

gives a rational parametrization of the unit circle and that

$$\mathcal{N}_2(t) = \frac{(1-t)^2 w_0 p_0 + 2t(1-t) w_1 p_1 + t^2 w_2 p_2}{(1-t)^2 w_0 + 2t(1-t) w_1 + t^2 w_2}.$$

can be used to parameterize a circular arc. Use the rational parametrization to obtain weights w_0, w_1, w_2 and control points p_0, p_1, p_2 such that $\mathcal{N}_2(t)$ parameterizes the first quarter of the unit circle for $t \in [0, 1]$.

Assignment 35 Consider a rational quadratic Bézier curve $\mathcal{B}$ with control points p_0, p_1, p_2 and weights $w_0 = w_2 := 1$ and $w_1 := w$ for some $w \in \mathbb{R}^+$. Let M be the mid-point of p_0 and p_2. Prove, possibly after simplifying matters by applying a suitable rigid motion, that $X := \mathcal{B}(1/2)$ lies on the line through M and p_1 such that

$$\frac{\|M - X\|}{\|M - P_1\|} = \frac{w}{1+w}.$$

Assignment 36 Let $\mathcal{N}$ be a NURBS curve of degree one defined by $n+1$ control points $p_0, p_1, \ldots, p_n$ and weights $w_0, \ldots, w_n \in \mathbb{R}^+$, over the clamped knot vector $\tau := (t_0, t_1, \ldots, t_{n+2})$, where all knots except t_0, t_1, t_n, t_{n+1} are distinct. Argue (without resorting to a convex-hull property, such as Lem. 159) that $\mathcal{N}$ equals the control polygon $(p_0, p_1, \ldots, p_n)$, irrelevant of the specific values of $w_0, \ldots, w_n$.

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PS Geometric Modeling
Homework Assignment Sheet X (Due 18-Dec-2020)

Assignment 37 Consider a rational quadratic Bézier curve $\mathcal{B}$ with control points p_0, p_1, p_2 and weights $w_0 = w_2 := 1$ and $w_1 := \|p_0 - p_2\|/(2\|p_0 - p_1\|)$. Prove that $\mathcal{B}$ models a circular arc if $\|p_0 - p_1\| = \|p_2 - p_1\|$.

Assignment 38 Consider the rational parametrization of the unit circle as given in Assignment 34. Which point do you get for $t := 1/2$? What is the speed at (a) the start point of the quarter circle, (b) the end point of the quarter circle, (c) for $t := 1/2$? Compare these results to what one obtains by selecting the weights according to Lemma 163 of the lecture slides. (You are welcome to use Mathematica or a similar tool to do the math.)

Assignment 39 Let $\mathcal{L}$ be the line segment that starts in the point a and ends in the point b, with a, b as specified below. Furthermore, let $\mathcal{N}$ be the rational quadratic NURBS curve with $\tau := (0, 0, 0, 1, 1, 1)$, control points p_0, p_1, p_2 and weights $w_0 := 1, w_1 := \frac{\sqrt{2}}{2}, w_2 := 1$.

$$a := \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad b := \begin{pmatrix} -1 \\ 0 \end{pmatrix} \quad p_0 := \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad p_1 := \begin{pmatrix} 0 \\ 2 \end{pmatrix} \quad p_2 := \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

Compute the Fréchet distance and the symmetric Hausdorff distance between $\mathcal{L}$ and $\mathcal{N}$.

Assignment 40 Specify three (families of) pairs of curves such that (1) their Hausdorff distance is small, (2) their Fréchet distance is large, and (3) a human would typically not regard them as similar. You do not need to write down mathematical definitions of each family of curves but it should be clear how one would modify your schemes in order to increase ratio of the Fréchet and the Hausdorff.