PS Geometric Modeling: Homework Assignment Sheet 1 for 2025-03-21

Assignment 1.1 Solve the following equation for $x \in \mathbb{R}$:

$$\frac{x+1}{|2x-3|} = x+2$$

Assignment 1.2 Consider $f, g : \mathbb{N} \to \mathbb{R}$ with $f(n) := \log n + 2n^2 + 2n + 4$ and $g(n) := n^2 + \sqrt{n}$. Which of the following claims is correct?

a) $f \in O(g)$, b) $f \in \Theta(g)$, c) $f \in \Omega(g)$, d) $f \in o(g)$

Assignment 1.3 For $g : \mathbb{N} \to \mathbb{R}^+$ we define

$$O(g) \;:=\; \left\{ f : \mathbb{N} \to \mathbb{R}^+ \mid \quad \exists n_0 \in \mathbb{N} \quad \exists c \in \mathbb{R}^+ \quad \forall n > n_0 \;:\quad f(n) \leqslant c\, g(n) \right\}$$

and

$$U(g) \;:=\; \left\{ f : \mathbb{N} \to \mathbb{R}^+ \mid \quad \exists c \in \mathbb{R}_0^+ \quad \lim_{n\to\infty} \frac{f(n)}{g(n)} = c \right\}.$$

Which set-theoretic relation holds between $O(g)$ and $U(g)$?

Assignment 1.4 Use induction to prove

$$\sum_{i=0}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

for all $n \in \mathbb{N}_0$. Please make sure to formulate a clean and mathematically sound proof!

Assignment 1.5 We define $s \colon \mathbb{N} \to \mathbb{Q}$ as follows

$$s_n := \begin{cases} \frac{1}{2}\left(s_{n-1} + \frac{5}{s_{n-1}}\right) & \text{if } n \geqslant 2, \\ 5 & \text{if } n = 1. \end{cases}$$

Your task is to determine $\min\{s_n \colon n \in \mathbb{N}\}$ and $\inf\{s_n : n \in \mathbb{N}\}$.

PS Geometric Modeling: Homework Assignment Sheet 2 for 2025-03-28

Assignment 2.1 Use induction to prove that

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

for all $n \in \mathbb{N}_0$. Please make sure to formulate a clean and mathematically sound proof!

Assignment 2.2 In how many zeros does 1000! end? (You should analyse 1000! and come up with a counting argument without determining the exact value of 1000!; please do not simply use Mathematica or a similar tool to compute it.)

Assignment 2.3 Prove that

$$\sum_{k=0}^{\lceil n/2 \rceil - 1} \binom{n}{2k+1} = 2^{n-1} \qquad \text{and} \qquad \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} = 2^{n-1}$$

for all $n \in \mathbb{N}$.

Assignment 2.4 Somebody claims that 1 and 4 are roots of the polynomal $p \in \mathbb{R}[x]$, with $p := x^5 - 10x^4 + 35x^3 - 50x^2 + 24x$. Verify this claim and compute the other roots of p. (You can either divide p appropriately, or resort to Vieta's rule and set up appropriate equations to obtain the coefficients of the remainder.)

Assignment 2.5 Prove explicitly: If $\nu_1, \ldots, \nu_n$ form a basis of the vector space V over $\mathbb{R}$, then for all $\nu \in V$ exist uniquely determined $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ such that $\nu = \lambda_1 \nu_1 + \lambda_2 \nu_2 + \cdots + \lambda_n \nu_n$.

PS Geometric Modeling: Homework Assignment Sheet 3 for 2025-04-04

Assignment 3.1 Compute $f_x(0,0)$ for the function $f\colon R \times R \to \mathbb{R}$, with

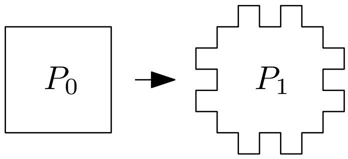
$$f(x,y) := \begin{cases} (x^2+y^2)\sin\left(\frac{1}{\sqrt{x^2+y^2}}\right) & \text{if } \ (x,y) \neq (0,0), \\ 0 & \text{if } \ (x,y) = (0,0). \end{cases}$$

Assignment 3.2 Prove that $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ with

$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

has partial derivatives on all of $\mathbb{R}^2$, can be differentiated on $\mathbb{R}^2 \backslash \{(0,0)\}$, but is not continuous at $(0,0)$.

Assignment 3.3 Let P_0 be a square of edge length 1. Now suppose that we generate a closed polygon P_1 by adding two small squares per edge of P_0 such that all edges of P_1 have one fifth of the length of the edges of P_0, as shown in the figure below. Inductively we generate P_{n+1} from P_n by adding two small squares on all of the edges of P_n (towards the outside of P_n) such that all edges of P_{n+1} have one fifth of the length of the edges of P_n.



What is the perimeter and the area of P_n? a tight upper bound on the area of P_n as n becomes arbitrarily large?

Assignment 3.4 Consider the infinite family of non-vertical lines (in $\mathbb{R}^2$) which all go through the point $(-1,0)$. Every such line is uniquely characterized by specifying a second point $(0,t)$ on the y-axis through which it passes. Your task is to parameterize the unit circle (except for the point $(-1,0)$) relative to the parameter t.

PS Geometric Modeling: Homework Assignment Sheet 4 for 2025-04-11

Assignment 4.1 Find a reparametrization β of $\gamma\colon \mathbb{R}_0^+ \to \mathbb{R}^3$, with $\gamma(t) := (a\cos t, a\sin t, bt)$ for $a, b \in \mathbb{R}^+$, which has unit speed. What is the arc length of β in dependence on the parameter t?

Assignment 4.2 Let $\gamma : I \to \mathbb{R}^3$ be a C^2 curve that is regular of order two, and define $T(t), N(t), B(t)$ as follows:

$T(t) := \frac{\gamma'(t)}{\lVert\gamma'(t)\rVert}$	unit tangent;
$N(t) := \frac{T'(t)}{\lVert T'(t)\rVert}$	unit (principal) normal;
$B(t) := T(t) \times N(t)$	unit binormal.

Prove that, for all $t \in I$,

- $N(t)$ is normal to $T(t)$, and
- $B(t)$ is a unit vector.

Assignment 4.3 Let $\gamma : I \to \mathbb{R}^3$ be a C^3-curve at unit speed that is of order two. Prove that $\lVert\gamma'(t) \times \gamma''(t)\rVert = \lVert\gamma''(t)\rVert$ for all $t \in I$.

Assignment 4.4 Find parametrizations of two curves that join in a C^1- and curvature-continuous fashion but that are not C^2-continuous.

Assignment 4.5 Determine the equation of the plane that is tangent to the graph (surface) of $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, with $f(x, y) := e^{-x^2-y^2}$, for $x := 1$ and $y := -1$.

PS Geometric Modeling: Homework Assignment Sheet 5 for 2025-05-09

Assignment 5.1 Prove directly — without resorting to (the proofs of) Lem. 97, Lem. 98 or Thm. 99! — that $B_{0,2}(x)$, $B_{1,2}(x)$ and $B_{2,2}(x)$ are linearly independent.

Assignment 5.2 Let $n \in \mathbb{N}$ and $k \in \mathbb{N}_0$ with $k \leqslant n$ be arbitrary but fixed. Prove that $B_{k,n}(x)$ has one global maximum within $[0,1]$, which occurs for $x := \frac{k}{n}$.

Assignment 5.3 Compute the second derivative of $B_{1,2}(x) + 2B_{2,2}(x)$. (You are welcome to use the formula stated in Cor. 96 for $B'_{k,n}(x)$, but you shall not use the formula for $B''_{k,n}(x)$.)

Assignment 5.4 Let p_0, p_1, p_2 be the control points of a quadratic Bézier curve $\mathcal{B}$. Let M be the mid-point of p_0 and p_2. Prove that $\mathcal{B}(1/2)$ lies on the line through M and p_1.

Assignment 5.5 Let $p_0, p_1, \ldots, p_n$ be a set of control points, and let $\mathcal{B}$ be the degree-n Bézier curve that is defined by $p_0, p_1, \ldots, p_n$. Furthermore, let $\mathcal{B}^\star$ and $\mathcal{B}^{\star\star}$ be the Bézier curves of degree $n-1$ that are defined by $p_0, p_1, \ldots, p_{n-1}$ and $p_1, p_2, \ldots, p_n$, respectively. Does the following equality hold for all $t \in [0,1]$?

$$\mathcal{B}(t) = (1-t)\mathcal{B}^\star(t) + t\mathcal{B}^{\star\star}(t)$$

PS Geometric Modeling: Homework Assignment Sheet 6 for 2025-05-16

Assignment 6.1 Consider the curve $\alpha\colon \mathbb{R} \to \mathbb{R}^2$ with $\alpha(t) := (3t + 3t^2, 1 + 4t^3)$ and compute a Bézier curve $\mathcal{B}$ such that $\mathcal{B} = \alpha|_{[0,1]}$.

Assignment 6.2 Let $n \in \mathbb{N}$ and consider a Bézier curve $\mathcal{B}\colon [0,1] \to \mathbb{R}^2$ of degree n with control points $p_0, p_1 \ldots, p_n$. Suppose that you used de Casteljau's algorithm to evaluate $\mathcal{B}(t^*)$ for some $t^* \in [0,1]$. How could you have used these computations to obtain $\mathcal{B}'(t^*)$ with very little computational overhead?

Assignment 6.3 Consider a cubic Bézier curve $\mathcal{B}\colon [0,1] \to \mathbb{R}^2$ and determine control points $p_0, p_1, p_2, p_3 \in \mathbb{R}^2$ such that

$$\mathcal{B}(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \mathcal{B}(1/4) = \mathcal{B}(3/4) \qquad \mathcal{B}(1) = \begin{pmatrix} 9 \\ 0 \end{pmatrix}$$

and such that it intersects itself at $\mathcal{B}(1/4) = \mathcal{B}(3/4)$ orthogonally. (You are welcome to use Mathematica (or a similar tool) for solving a system of linear equations; it's not necessary, though.)

Assignment 6.4 Prove or disprove the following claim: Given numbers $y_0, v_0, y_1, v_1 \in \mathbb{R}$, there exists a unique polynomial $P\colon \mathbb{R} \to \mathbb{R}$ of degree at most three such that

$$P(0) = y_0 \quad \text{and} \quad P'(0) = v_0 \quad \text{and} \quad P(1) = y_1 \quad \text{and} \quad P'(1) = v_1.$$

Please try to apply knowledge gained in the lecture rather than brute-force calculations.

Assignment 6.5 Consider the Bézier surface

$$\mathcal{S}(u,v) := \sum_{i=0}^{n} \sum_{j=0}^{m} B_{i,n}(u) B_{j,m}(v) p_{i,j} \qquad \text{for } (u,v) \in [0,1] \times [0,1],$$

with the control points $p_{i,j}$ for $0 \leqslant i \leqslant n$ and $0 \leqslant j \leqslant m$. Resort to the definition of a tangent plane (Def. 74) to verify that the tangent plane in $\mathcal{S}(1,1)$ is spanned by the vectors $p_{n,m} - p_{n-1,m}$ and $p_{n,m} - p_{n,m-1}$.