Computing and Testing Small Connectivity in Near-Linear Time and Queries via Fast Local Cut Algorithms

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University of Salzburg

Joint work with Danupon Nanongkai, Thatchaphol Saranurak, Liu Yang, and Sorrachai Yingchareonthawornchai

Workshop: Recent Trends in Theoretical Computer Science

Edge and Vertex Connectivity

Edge connectivity λ/vertex connectivity κ

Minimum number of edges/vertices to remove in order to make the graph not strongly connected

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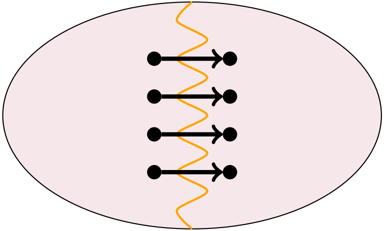

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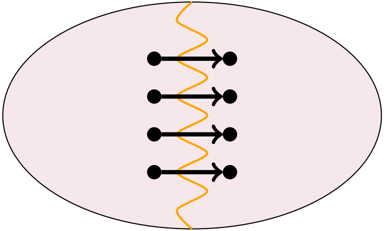

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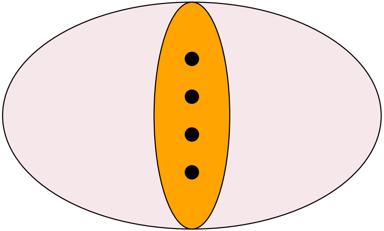

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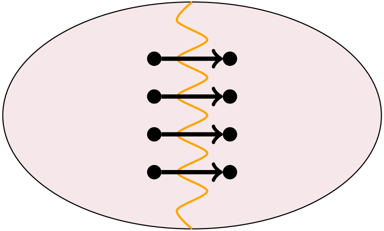

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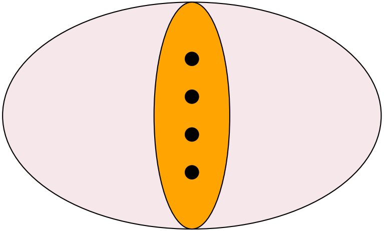

Motivation:

- Fundamental graph-theoretic notion
- Applications: Reliability analysis, community detection

State of the Art and Results

Vertex connectivity in directed graphs:

Running time	Deterministic	Reference
$\tilde{O}(n^{2.373} + n\kappa^{2.373})$	no	[Cheriyan/Reif '92]
$\tilde{O}(mn)$	no	[Henzinger et al. '96]
$O(mn + \kappa mn^{3/4})$	yes	[Gabow '00]
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$\tilde{O}(\kappa m^{4/3})$	no	[Nanongkai et al. '19]
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Improvements also for finding k-edge connected subgraphs [Chechik et al. '17]

Property Testing Results

Algorithm needs to distinguish between graphs that are k-connected and graphs that are ϵ-**far** from being k-connected (cannot be made k-connected by changing an ϵ-fraction of the edges). Want to minimize the number of **edge queries** to the graph.

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Similar improvements for graphs of unbounded degree (w.r.t. avg. degree)

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There is a local procedure that, given a seed vertex s, a target cut size k and a target volume Δ runs in time $O(k^2\Delta)$, and returns as follows:

1. *If s is contained in an ℓ-out component of volume $\leq \Delta$ for $\ell \leq k$, then it returns an ℓ-out component of volume $\leq 3k\Delta$ with probability at least $\frac{1}{2}$*
2. *Otherwise, it might return a k-out-component or $\perp$*

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Prior work:

- "Local" version of Karger's algorithm [Goldreich/Ron '02]
- Exponential time [Orenstein/Ron '11] [Chechik et al. '17]
- Local flow techniques [Nanongkai/Saranurak/Yingchareonthawornchai '19]

Randomization of Augmenting-Path Idea [Chechik et al. '17]

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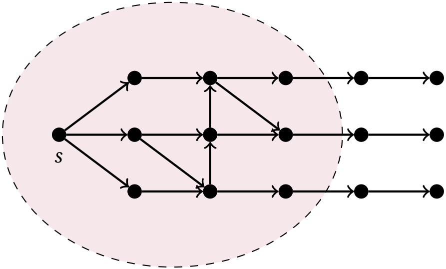

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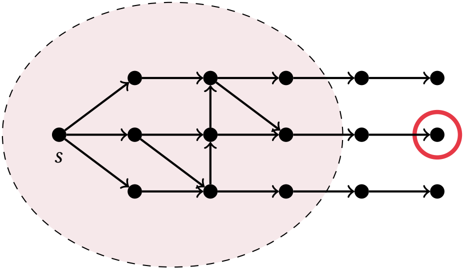

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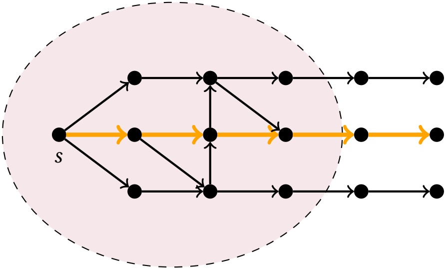

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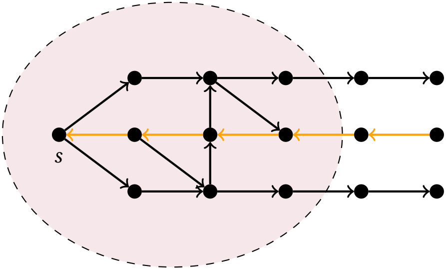

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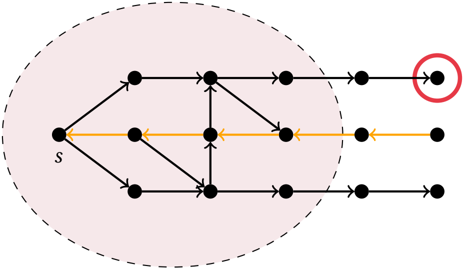

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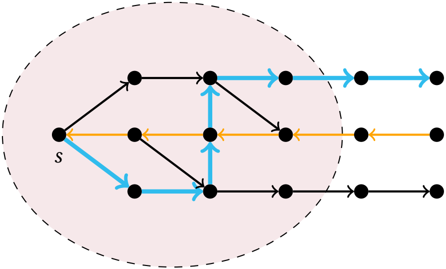

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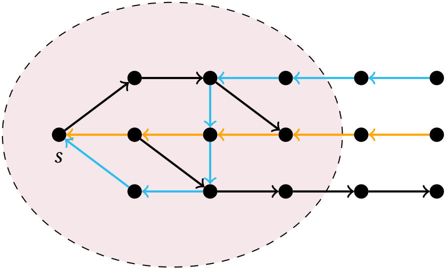

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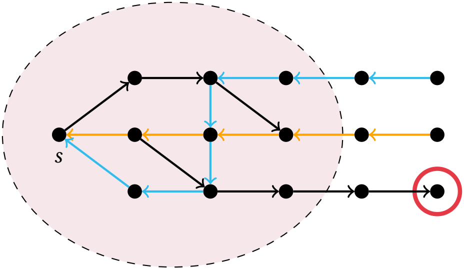

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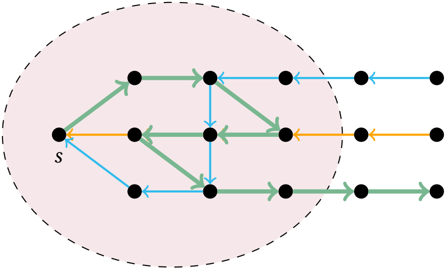

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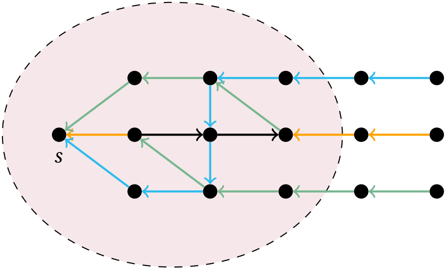

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Claim 1 [Chechik et al. '17]

Let $U \subseteq V$ contain s, let $t \in V$, and reverse the edges on a path from s to t.

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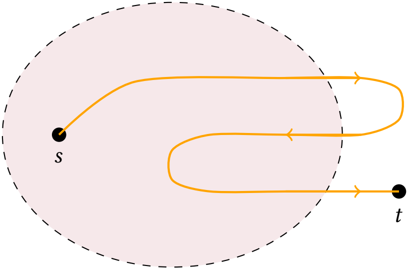

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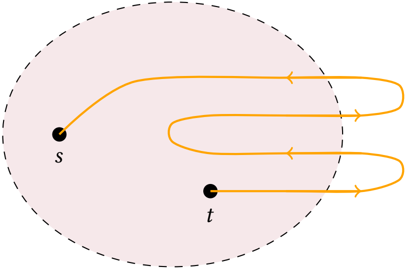

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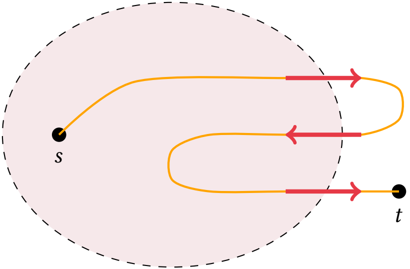

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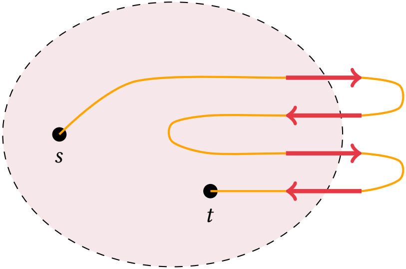

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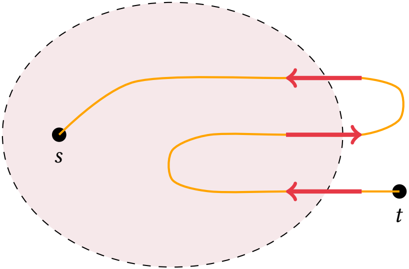

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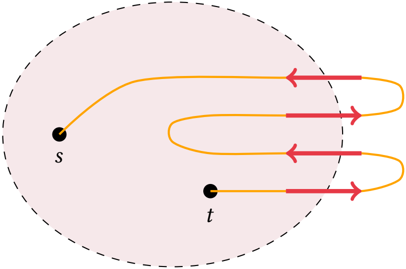

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If the procedure returns a set of vertices U in iteration $\ell + 1$, then U is an ℓ-out-component with $\mathrm{vol}(U) \leq 2k\Delta + \ell \leq 3k\Delta$.

Idea: For component found by DFS, number of out-edges reduces by at most one in each iteration

Claim 3

If there is an ℓ-out-component C of volume $\leq \Delta$ containing s for $\ell \leq k$, then the procedure returns an ℓ-out-component with probability $\geq \frac{1}{2}$.

Proof

- Algorithm succeeds if in first k iterations always tail of sampled edge outside of component C (known to exist)
- $\mathrm{vol}(C) \leq \Delta$ and DFS processes $= 2k\Delta$ many edges
- Tail of sampled edge will lie inside of C with probability $\leq \frac{1}{2k}$

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- By Union Bound: algorithms fails with probability $\leq \frac{1}{2}$

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- Significant progress for fundamental graph problems
- Local procedure was pivotal to better time/query complexities
  Exponential improvement: from $O(2^{O(k)}\Delta)$ [Chechik et al. '17] to $O(k^2\Delta)$ at the cost of randomization

Thank you!