Single-Source Shortest Paths: Towards Optimality

Sebastian Forster

Department of Computer Sciences
University of Salzburg, Austria
Previously known as S. Krinninger

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joint works with

Ruben Becker
Monika Henzinger
Andreas Karrenbauer
Christoph Lenzen
Danupon Nanongkai
Problem Definition
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Goal: Compute shortest paths from a source node $s$ to all other nodes
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- Not fully understood in PRAM model
Context

**Goal:** Compute shortest paths from a source node \( s \) to all other nodes

**How can this be an open problem??**

- (Nearly) optimal solutions known in RAM model
- Not fully understood in CONGEST model
- Not fully understood in PRAM model
- To be fair: non-negative weights also not fully understood in RAM model
CONGEST Model

**Idea:** Measure amount of communication for network to compute result
Running time = #communication rounds
CONGEST Model

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**Model definition:**
- Processors with unique IDs modeled as nodes
- Synchronous rounds (global clock)
- In each round, every node sends (at most) one message to each neighbor
- Message size $O(\log n)$
- Unlimited internal computation between rounds
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- Communication network: unweighted undirected graph of diameter $D$
- Edges are “annotated” with (non-negative) weights and directions
- Weights represent costs (not time)
- This talk: integer edge weights bounded by $n^{O(1)}$
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**Distributed problem statement:**
- Initial knowledge: incident edges, source
- Terminal knowledge: distance to the source, parent on shortest path tree
Unweighted Graphs: BFS

Breadth-first search tree can be computed in \( O(D) \) rounds.

Our goal: efficient algorithms for weighted graphs
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**Our goal:** efficient algorithms for weighted graphs
Known Results

Exact SSSP:

\[ O(n) \]
\[ \tilde{O}(n^{2/3}D^{1/3} + n^{5/6}) \]

Bellman-Ford

[Elkin ’17]

\[ 1 \leq 1/\log^{O(1)} n \]

Known Results

**Exact SSSP:**

\[ O(n) \]
\[ \tilde{O}(n\sqrt{D}) \]
\[ \tilde{O}(n^{3/4}D^{1/4}) \]
\[ \tilde{O}(n^{3/4} + o(1) + \min\{n^{3/4}D^{1/6}, n^{6/7}\} + D) \]

Bellman-Ford

[Elkin ’17]

[Ghaffari/Li ’18]

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\[ 1 - \epsilon \geq 1/\log^{O(1)} n \]
Known Results

Exact SSSP:

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Bellman-Ford  

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$\tilde{O}(\sqrt{nD})$  

[F/Nanongkai]

$\tilde{O}(\sqrt{nD^{1/4}} + n^{3/5} + D)$  

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(1 + \epsilon)-approximate SSSP:

\[ \tilde{O}((\sqrt{nD^{1/4}} + D)/\epsilon^{O(1)}) \]

\[ \tilde{O}((\sqrt{n} + D)n^{o(1)}) \]

\[ \tilde{O}((\sqrt{n} + D)/\epsilon^{O(1)}) \]

[Nanongkai ’14]

[Henzinger/K/Nanongkai ’16]

[Becker/Karrenbauer/K/Lenzen ’17]

\(^1\epsilon \geq 1/\log^{O(1)} n\)
Known Results

**Exact SSSP:**

\[ O(n) \]
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\[ \tilde{O}(n^{3/4+o(1)} + \min\{n^{3/4}D^{1/6}, n^{6/7}\} + D) \]
\[ \tilde{O}(\sqrt{nD}) \]
\[ \tilde{O}(\sqrt{nD}^{1/4} + n^{3/5} + D) \]

Bellman-Ford
[Elkin ’17]

\[ \tilde{O}(\sqrt{nD}^{1/4} + D) / \epsilon^{O(1)} \]
\[ \tilde{O}((\sqrt{n} + D)n^o(1)) \]
\[ \tilde{O}((\sqrt{n} + D) / \epsilon^{O(1)}) \]

[Nanongkai ’14]
[Henzinger/K/Nanongkai ’16]
[Becker/Karrenbauer/K/Lenzen ’17]

**Common Lower Bound:**

\[ \tilde{\Omega}(\sqrt{n} + D) \]

[Peleg/Rubinovich ’99]
[Elkin ’04]
[Das Sarma et al. ’11]
More Related Work

Approximation Algorithms:

- [Lenzen/Patt-Shamir ’13]
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All-Pairs Shortest Paths and $k$-Source Shortest Paths:
- [Holzer/Wattenhofer ’12]
- [Elkin/Neiman ’16]
- [Huang/Nanongkai/Saranurak ’17]
- [Agarwal/Ramachandran/King/Pontecorvi ’18]
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Congested Clique:
- [Censor-Hillel et al. ’15]
- [Holzer/Pinsker ’15]
Basic Tools
Lemma

Suppose \( k \) pieces of information (of size \( O(\log n) \) each) are distributed among the nodes of the network. All this information can be made known to all nodes in \( O(k + D) \) rounds.

Need to respect bounded message size!
Broadcasting

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Suppose \( k \) pieces of information (of size \( O(\log n) \) each) are distributed among the nodes of the network. All this information can be made known to all nodes in \( O(k + D) \) rounds.

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Algorithm:

1. Compute BFS tree (from arbitrary root)
2. Aggregate information at root bottom up Queue of outgoing messages at each node
3. Distribute information from root top down Send one piece at a time

“Pipelining”
Broadcasting

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1. Compute BFS tree (from arbitrary root)
   Time: $O(D)$

2. Aggregate information at root bottom up
   Queue of outgoing messages at each node
   Time: $O(k + D)$

3. Distribute information from root top down
   Send one piece at a time
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Bellman-Ford

Algorithm:

1. Initialize $\delta_0(s) = 0$ and $\delta(v) \neq 0$ for $v \neq s$
2. In round $i$, set $\delta_i(v) = \min_{(u,v) \in E}(\delta_{i-1}(u) + w(u, v))$
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Can compute shortest paths from given source in $O(n)$ rounds
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Lemma

*Can compute shortest paths from given source in $O(n)$ rounds*

Fine-grained analysis: After $h$ rounds, algorithm has computed shortest $h$-hop paths (shortest among all paths with a “budget” of $h$ edges)
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Lemma

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Intuition

SSSP is easy if shortest path has only few edges (hops)!
Hopsets

Definition ([Cohen ’00])

An $(h, \varepsilon)$-hopset is a set of weighted edges $F$ such that, for every pair of nodes $u$ and $v$, there is a path from $u$ to $v$ with at most $h$ edges of weight at most $(1 + \varepsilon) \text{dist}_G(u, v)$ in $G \cup F$. 

Attention: Hopset edges cannot literally be “added” to network!
Hopsets

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**Observation**

Given \((h, \epsilon)\)-hopset, \(h\)-hop shortest paths provide \((1 + \epsilon)\)-approximation
## Hopsets

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**Attention:** Hopset edges cannot literally be “added” to network!
Skeleton Graph: Intuition
Skeleton Graph

Randomized skeleton $H$:

1. Sample $\tilde{O}(n/h)$ skeleton nodes uniformly at random (+ source $s$)
2. Set $w_H(x, y) = \text{dist}^h_G(x, y)$ ($h$-hop distance)
Skeleton Graph

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Lemma ([Klein/Subramanian ’97])

Skeleton is an exact $(\tilde{O}(n/h + h), 0)$-hopset with high probability.
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Proof of hopset property:
**Skeleton Graph**

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**Proof of hopset property:**

![Diagram](image)
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Proof of hopset property:

![Diagram showing the proof of hopset property](image-url)
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Proof of hopset property:
Skeleton Shortcuts: Intuition
Skeleton Shortcuts

1. Suppose we could compute SSSP on skeleton $H$
2. Shortcut edges $F$ from $s$ to skeleton nodes: $w_F(s, x) = \text{dist}_H(s, x)$
Skeleton Shortcuts

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2. Shortcut edges $F$ from $s$ to skeleton nodes: $w_F(s, x) = \text{dist}_H(s, x)$

Observation

Shortcuts $F$ are an exact source-wise $(h, 0)$-hopset with high probability.
Suppose we could compute SSSP on skeleton $H$.

Shortcut edges $F$ from $s$ to skeleton nodes: $w_F(s, x) = \text{dist}_H(s, x)$

**Observation**

Shortcuts $F$ are an exact source-wise $(h, 0)$-hopset with high probability.

**Recall proof:**

![Diagram showing the hopset and shortcuts](image)
Skeleton Shortcuts

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**Skeleton Shortcuts**

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**Observation**

Shortcuts $F$ are an exact source-wise $(h, 0)$-hopset with high probability.

**Recall proof:**

**Good news:**

- Cannot literally “add” shortcuts to network, but can run Bellman-Ford on $G \cup F$
Suppose we could compute SSSP on skeleton $H$
Shortcut edges $F$ from $s$ to skeleton nodes: $w_F(s, x) = \text{dist}_H(s, x)$

**Observation**
Shortcuts $F$ are an exact *source-wise* $(h, 0)$-hopset with high probability.

**Recall proof:**

**Good news:**
- Cannot literally “add” shortcuts to network, but can run Bellman-Ford on $G \cup F$
- Only first iteration uses shortcut edges of $F$
- If each skeleton node knows shortcut to $s$, simulate first iteration in $O(D)$ rounds
Suppose we could compute SSSP on skeleton $H$

Shortcut edges $F$ from $s$ to skeleton nodes: $w_F(s, x) = \text{dist}_H(s, x)$

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**Good news:**

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A First Idea

Algorithm 1:

1. Determine skeleton nodes: random sample of $\tilde{O}(n/h)$ nodes + s
   (repeat sampling if too large)

2. Compute $h$-hop distances from all skeleton nodes
   (such that $\text{dist}_G^h(x, v)$ is known to $v$)

3. Make skeleton known to every node

4. Determine set of shortcut edges $F$
   (Internally compute SSSP on skeleton $H$ for every node)

5. Compute $h$-hop distances from $s$ in $G \cup F$
   ($h$ Bellman-Ford iterations)
A First Idea

Algorithm 1:

1. Determine skeleton nodes: random sample of $\tilde{O}(n/h)$ nodes + s
   (repeat sampling if too large)
   \textbf{Time:} $O(D)$

2. Compute $h$-hop distances from all skeleton nodes
   (such that $\text{dist}_G^h(x, v)$ is known to $v$)
   \textbf{Time:} $\tilde{O}(h \cdot n/h) = \tilde{O}(n)$ (sequential)

3. Make skeleton known to every node
   \textbf{Time:} $O(n^2/h^2 + D)$

4. Determine set of shortcut edges $F$
   (Internally compute SSSP on skeleton $H$ for every node)
   \textbf{Time:} 0

5. Compute $h$-hop distances from $s$ in $G \cup F$
   ($h$ Bellman-Ford iterations)
   \textbf{Time:} $O(h)$
A First Idea

Algorithm 1:

1. Determine skeleton nodes: random sample of \( \tilde{O}(n/h) \) nodes + s (repeat sampling if too large)
   Time: \( O(D) \)

2. Compute \( h \)-hop distances from all skeleton nodes
   (such that \( \text{dist}^h_G(x, v) \) is known to \( v \))
   Time: \( \tilde{O}(h \cdot n/h) = \tilde{O}(n) \) (sequential)

3. Make skeleton known to every node
   Time: \( O(n^2/h^2 + D) \)

4. Determine set of shortcut edges \( F \)
   (Internally compute SSSP on skeleton \( H \) for every node)
   Time: 0

5. Compute \( h \)-hop distances from \( s \) in \( G \cup F \)
   (\( h \) Bellman-Ford iterations)
   Time: \( O(h) \)
Multiple Bounded-Hop Distances

**Goal:** Run $\tilde{O}(n/h)$ instances of Bellman-Ford ($h$ iterations) “in parallel”
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**Obstacle:**
- In each instance, every node sends to all its neighbors
- One iteration in all instances: up to $\tilde{O}(n/h)$ messages over each edge
Multiple Bounded-Hop Distances

**Goal:** Run \( \tilde{O}(n/h) \) instances of Bellman-Ford (\( h \) iterations) “in parallel”

**Obstacle:**
- In each instance, every node sends to all its neighbors
- One iteration in all instances: up to \( \tilde{O}(n/h) \) messages over each edge
- Bandwidth only allows one message
- Could simulate sending of \( \tilde{O}(n/h) \) messages in \( \tilde{O}(n/h) \) rounds
Multiple Bounded-Hop Distances

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Alternative to Bellman-Ford: “Weighted BFS”
- Replace each weighted edge $e$ by path of $w(e)$ unweighted edges
Multiple Bounded-Hop Distances

**Goal:** Run $\tilde{O}(n/h)$ instances of Bellman-Ford ($h$ iterations) “in parallel”

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Alternative to Bellman-Ford: “**Weighted BFS**”
- Replace each weighted edge $e$ by path of $w(e)$ unweighted edges
- Replacement can be simulated in BFS computation
- Can compute shortest paths of weight $\leq L$ in time $O(L)$
Multiple Bounded-Hop Distances

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Obstacle:
- In each instance, every node sends to all its neighbors
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Alternative to Bellman-Ford: “Weighted BFS”
- Replace each weighted edge $e$ by path of $w(e)$ unweighted edges
- Replacement can be simulated in BFS computation
- Can compute shortest paths of weight $\leq L$ in time $O(L)$
- Bandwidth-friendly: at most one message per node
- Pseudopolynomial: $h$-hop shortest paths in time $O(hW_{\max})$
Approximate Bounded-Hop Distances

Weight rounding technique: [Klein/Subramanian ’97]

- Round up weights to multiples of $\varphi$

Lemma ([Nanongkai ’14])

Can compute $(1 + \varphi)$-approximate $h$-hop shortest paths from given source in $\tilde{O}(h/\varphi)$ rounds such that each node sends $\tilde{O}(1/\varphi)$ messages.
Approximate Bounded-Hop Distances

**Weight rounding technique: [Klein/Subramanian ’97]**

- Round up weights to multiples of $\phi$
- Scale down rounded weights to integers
- Speed-up: shortest paths of weight $\leq L$ in time $O(L/\phi)$
Approximate Bounded-Hop Distances

**Weight rounding technique:** [Klein/Subramanian ’97]

- Round up weights to multiples of $\phi$
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- But: Each edge traversal gives additive error of $\phi$
Approximate Bounded-Hop Distances

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- Round up weights to multiples of $\varphi$
- Scale down rounded weights to integers
- Speed-up: shortest paths of weight $\leq L$ in time $O(L/\varphi)$
- But: Each edge traversal gives additive error of $\varphi$
- Choice of $\varphi_i = \epsilon 2^i / h$ deals with range $2^i \leq \text{dist}_h^i(s, v) \leq 2^i + 1$

Lemma ([Nanongkai ’14])

Can compute $(1 + \epsilon)$-approximate $h$-hop shortest paths from given source in $\tilde{O}(h/\epsilon)$ rounds such that each node sends $\tilde{O}(1/\epsilon)$ messages
Multiple Approximate Bounded-Hop Distances

Efficient parallelization: Random start delays [Leighton/Maggs/Rao ’94]

- For each skeleton node: random integer delay from 0 to $\tilde{O}(n/h)$
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**Lemma ([Nanongkai ’14])**

*Can compute $(1 + \epsilon)$-approximate skeleton of $\tilde{O}(n/h)$ nodes in time $\tilde{O}(h/\epsilon + n/h)$*
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**Remarks:**
- Alternative: Weight rounding + source detection [Lenzen/Peleg ’13]
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**Remarks:**

- Alternative: Weight rounding + source detection [Lenzen/Peleg ’13]
- Approximate skeleton is $(\tilde{O}(n/h + h), \epsilon)$ hopset
Refined Algorithm

**Algorithm 2:**

1. **Determine skeleton nodes:** random sample of $\tilde{O}(n/h)$ nodes + s (repeat sampling if too large)

2. **Compute $(1 + \epsilon)$-approximate $h$-hop distances from all skeleton nodes** (such that $\text{dist}_G^h(x, v)$ is known to $v$)

3. **Make skeleton known to every node**

4. **Determine set of shortcut edges $F$**  
   (Internally compute SSSP on skeleton $H$ for every node)

5. **Compute $h$-hop distances from $s$ in $G \cup F$**
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1. Determine skeleton nodes: random sample of $\tilde{O}(n/h)$ nodes + $s$ (repeat sampling if too large)
   *Time: $O(D)$*

2. Compute $(1 + \varepsilon)$-approximate $h$-hop distances from all skeleton nodes (such that $\text{dist}^h_G(x, v)$ is known to $v$)
   *Time: $\tilde{O}(h/\varepsilon + n/h)$*

3. Make skeleton known to every node
   *Time: $O(n^2/h^2 + D)$*

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   Time: $O(h)$

Theorem

Can compute $(1 + \varepsilon)$-approximate SSSP in time $\tilde{O}(n^{2/3}/\varepsilon + D)$ with $h = n^{2/3}$
Computing on Skeleton via Broadcast

**Goal:** Recurse on skeleton to improve efficiency
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**Obstacle:**
- Edges between skeleton nodes do not exist in communication network!
- How to run algorithm “on” skeleton?
Computing on Skeleton via Broadcast

**Goal:** Recurse on skeleton to improve efficiency

**Obstacle:**
- Edges between skeleton nodes do not exist in communication network!
- How to run algorithm “on” skeleton?

**Idea:** Simulate a round with total of $k$ messages on skeleton by making all messages global knowledge in time $O(k + D)$
Reduction to Blackboard model

Blackboard model:

- Communication in synchronized rounds
- Write messages on “blackboard” to make them global knowledge
- No congestion constraint, only total size of messages is relevant

Lemma ([Nanongkai ’14]): Any algorithm with $R(k)$ rounds and messages of total size $M(k)$ in blackboard model, can be simulated on skeleton of $k$ nodes in $\tilde{O}(M(k) + R(k)D)$ rounds in the CONGEST model.
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(Shared-memory clique??)
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Back to Our Algorithm

Algorithm 3:

1. Determine skeleton nodes: random sample of $\tilde{O}(n/h)$ nodes + s

2. Compute $(1 + \epsilon)$-approximate $h$-hop distances from all skeleton nodes

3. Compute $(1 + \epsilon)$-approximate shortest paths from s on skeleton
   Simulate Algorithm 2 with $R(k) = \tilde{O}(h'/\epsilon)$ and $M(k) = k^2/(h\epsilon)$ where $k = \tilde{O}(n/h)$.

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   Time: $O(n^2/(\epsilon h^2 h') + Dh'/\epsilon)$
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   Time: $O(h)$
Back to Our Algorithm

**Algorithm 3:**

1. **Determine skeleton nodes:** random sample of $\tilde{O}(n/h)$ nodes + $s$
   
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4. **Determine set of shortcut edges $F$**
   
   **Time:** $0$

5. **Compute $h$-hop distances from $s$ in $G \cup F$**
   
   **Time:** $O(h)$

**Theorem ([F/Nanongkai ’18])**

*Can compute $(1 + \varepsilon)$-approximate SSSP in time* $\tilde{O}((\sqrt{n}D^{1/4} + D)/\varepsilon)$ *with* $h = \sqrt{n}D^{1/4}$ *and* $h' = \sqrt{n}/D^{3/4}$
Exact SSSP
Scaling Approach

Two scaling techniques [Gabow ’85]:

1. **Bitwise scaling:** In each iteration read next bit of weights
2. **Recursive scaling:** Reduce maximum distance by potential transformation with approximate distances

Potential transformation:

$$w'(u, /v.alt) = w_G(u, /v.alt) + \hat{d}(s, u) - \hat{d}(s, /v.alt)$$

Does not change shortest paths

Solve recursively with weights $w'$: Maximum distance has halved!

But:

Want to keep edge weights non-negative

Additional constraint:

$$\hat{d}(s, /v.alt) \leq \hat{d}(s, u) + w_G(u, /v.alt)$$
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- Additional constraint: \( \hat{d}(s, v) \leq \hat{d}(s, u) + w_G(u, v) \)
Reduction

Theorem ([Klein/Subramanian ’97])

Suppose auxiliary algorithm computes distance estimate $\hat{d}(s, \cdot)$ such that

- For every node $v$: $\frac{1}{2} \cdot \text{dist}_G(s, v) \leq \hat{d}(s, v) \leq \text{dist}_G(s, v)$ (approximation)
- For every edge $(u, v)$: $\hat{d}(s, v) \leq \hat{d}(s, u) + w_G(u, v)$ (domination)

Then exact SSSP can be computed by calling auxiliary algorithm $O(\log(nW_{\text{max}}))$ times (+ bookkeeping work).

Our contribution:
Design suitable auxiliary algorithm
Leverage techniques from approximate SSSP
Careful design to satisfy domination constraint

Fine print:
Inherent dependence on $\log(W_{\text{max}})$ to bound maximum distance
Must solve directed problem
Must accept $0$-weight edges
→ Reduction to positive edge weights
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Auxiliary Algorithm

1. Determine skeleton nodes: random sample of $\tilde{O}(n/h)$ nodes + s

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3. Compute exact SSSP on skeleton

4. Determine set of shortcut edges $F$

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Auxiliary Algorithm

1. Determine skeleton nodes: random sample of $\tilde{O}(n/h)$ nodes + $s$
   Time: $O(D)$

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   Time: ???

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Theorem

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Proof of Domination

- Need to show: \( \text{dist}^h_{G \cup F}(s, v) \leq \text{dist}^h_{G \cup F}(s, u) + w_G(u, v) \)
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Proof idea:
- Shortest path in \( G \cup F \) has the following structure: at most one shortcut edge to skeleton node followed by a shortest path \( \pi \) in \( G \)
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- Following skeleton nodes with skeleton edges would be at least as cheap as following \( \pi \) (underestimated approximation!)
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- Shortcut edge in \( G \cup F \) to last skeleton node is as least as cheap
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- Reason: Triangle inequality for exact distances!
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- Need to show: $\text{dist}_{G \cup F}^h(s, v) \leq \text{dist}_{G \cup F}^h(s, u) + w_G(u, v)$
- We show that $\text{dist}_{G \cup F}^h(s, v) = \text{dist}_{G \cup F}(s, v)$
- Then domination follows from triangle inequality

**Proof idea:**
- Shortest path in $G \cup F$ has the following structure: at most one shortcut edge to skeleton node followed by a shortest path $\pi$ in $G$
- Subdivide $\pi$ into subsequent chunks of $h/2$ edges
- With high probability, each chunk contains a skeleton node
- Following skeleton nodes with skeleton edges would be at least as cheap as following $\pi$ (underestimated approximation!)
- Shortcut edge in $G \cup F$ to last skeleton node is as least as cheap
- Reason: Triangle inequality for exact distances!
- Now: remainder of $\pi$ has $< h$ edges
How to Solve on Skeleton

Recall: We need exact SSSP on skeleton to compute shortcuts
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Two Variants:

1. Dijkstra’s algorithm on skeleton

2. Recurse on skeleton using our new algorithm
How to Solve on Skeleton

Recall: We need *exact* SSSP on skeleton to compute shortcuts

Two Variants:

1. Dijkstra’s algorithm on skeleton
   - $\tilde{O}(n/h)$ iterations
   - Time $O(D)$ per iteration
   - Total running time: $\tilde{O}(\sqrt{nD})$

2. Recurse on skeleton using our new algorithm
How to Solve on Skeleton

Recall: We need exact SSSP on skeleton to compute shortcuts

Two Variants:

1. Dijkstra’s algorithm on skeleton
   - $\tilde{O}(n/h)$ iterations
   - Time $O(D)$ per iteration
   - Total running time: $\tilde{O}(\sqrt{nD})$

2. Recurse on skeleton using our new algorithm
   - Blackboard model:
     - $R(k) = \tilde{O}(h)$ rounds
     - $M(k) = \tilde{O}(nh + n^2/h)$ messages
   - Total running time: $\tilde{O}(\sqrt{nD}^{1/4} + n^{3/5} + D)$
Discussion: Implementation of Klein/Subramanian?

We borrow many ideas from PRAM algorithm of Klein and Subramanian.
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Main difference:
- Klein and Subramanian: Skeleton as hopset
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New trade-off for directed graphs in PRAM model:
- Klein and Subramanian: work $\tilde{O}(m\sqrt{n})$ and depth $\tilde{O}(\sqrt{n})$
- Our approach: work $\tilde{O}((n^3/h^3 + mh + mn/h))$ and depth $\tilde{O}(h)$
Faster Approximation
Broadcast Congested Clique

Model:
- Network topology is a clique
- In each round, every node sends one message to all its neighbors
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Any broadcast congested clique algorithm with $R(k)$ rounds can be simulated on skeleton of $k$ nodes in $O((k + D)R(k))$ rounds in the CONGEST model.
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Theorem ([Nanongkai ’14])

In directed graphs, can compute $(1 + \epsilon)$-approximate skeleton with $k = \tilde{O}(\sqrt{n})$ nodes in $\tilde{O}(\sqrt{n})$ rounds. The algorithm is correct with high probability.
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Theorem ([Henzinger/K/Nanongkai ’16])

In undirected graphs, can compute $(1 + \epsilon)$-approximate skeleton with $k = \tilde{O}(\sqrt{n})$ nodes deterministically in $\tilde{O}(\sqrt{n})$ rounds.
Fast Hopset Construction

**Theorem ([Henzinger/K/Nanongkai ’16])**

Can compute $(1 + \epsilon)$-approximate SSSP on undirected Broadcast Congested Clique deterministically in $n^{o(1)}$ rounds for any given $\epsilon \geq 1/\log^{O(1)}$. 

Recall: Given $(h, \epsilon)$-hopset, $(1 + \epsilon)$-approximate SSSP can be computed in $O(h)$ rounds.

Ideas:

Observation: distance oracle of [Thorup/Zwick '05] gives $(n^{o(1)}, \epsilon)$-hopset in undirected graphs [Bernstein '09].

Vanilla Thorup/Zwick already requires SSSP computation.

Iterative Approach: Bounded-hop SSSP allows hop reduction. Hopset is obtained after sufficiently many hop reductions.

Remarks:

Hopset lower bound indicates $n^{o(1)}$ barrier [Abboud/Bodwin/Pettie ’17].

Tight hopsets exist [Huang/Pettie ’17] [Elkin/Neiman ’17].
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Theorem ([Becker/Karrenbauer/K/Lenzen ’17])

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Linear Programming Formulation

**Primal:** minimize $\|Wx\|_1$ s.t. $Ax = b$

**Dual:** maximize $b^Ty$ s.t. $\|W^{-1}A^Ty\|_{\infty} \leq 1$
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- More general problem: Uncapacitated minimum-cost flow
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- Markov-style argument for finding approximate distances
Conclusion

Take-home message:

- Wide array of techniques
- Approximate SSSP with nearly tight running time
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2. Find deterministic sublinear exact algorithm
3. Is $\tilde{O}(\sqrt{n})$ rounds tight on Broadcast Congested Clique?
Thank you!

slides: https://www.cosy.sbg.ac.at/~forster/