Time-Space Trade-offs in Population Protocols

2017

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Overview

- What are population protocols?
- What is leader election?
- What is the majority problem?
- Overview of the Paper
- Lower Bound for majority and leader election
- New algorithms for majority and leader election
  - Lottery Election (detailed)
  - Split-Join Majority (overview)

Population Protocols

- Set of $n \geq 2$ agents
- Each executing deterministic state machine
- State from a finite set Λ_n, might depend on n
- Transition function $\delta_n : \Lambda_n$ x Λ_n → Λ_n x Λ_n
- Output function $\gamma_n : \Lambda_n$ → O
- Pairs of agents are chosen uniformly at random
- Each agent updates state according to transition function δ_n
- Result can be checked with output function

Leader Election

- All agents start in the same initial state
- Output set O is $\{Win, Lose\}$
- Goal: One agent has Output Win, the rest has $Lose$

Majority Problem

- Two initial states A_n, B_n
- Output set is $\{Win_a, Win_b\}$
- Goal: Output of every agent should correspond to majority of initial state
- If A_n, B_n are split 50/50, output is arbitrary

The Paper

- Title: Time-Space Trade-Offs in Population Protocols (2017)
- Authors:
  - Alistarh (ETH Zürich)
  - Aspnes (Yale)
  - Eisenstat (Google)
  - Gelashvili (MIT)
  - Rivest (MIT)
- Trade-off between number of states and running time
- New and improved algorithms for majority and leader election

The Paper

Problem	Type	Expected Time Bound	Number of States	Reference
Exact Majority $\epsilon = 1/n$	Algorithm	$O(n \log n)$	4	[DV12, MNRS14]
	Algorithm	$O(\log^2 n)$	$\Theta(n)$	[AGV15]
	Lower Bound	$\Omega(n)$	≤ 4	[AGV15]
	Lower Bound	$\Omega(\log n)$	any	[AGV15]
Leader Election	Algorithm	$O(\log^3 n)$	$O(\log^3 n)$	[AG15]
	Lower Bound	$\Omega(n)$	$O(1)$	[DS15]
Exact Majority Leader Election	Lower Bound	$\Omega(n/\text{polylog} n)$	$< 1/2 \log \log n$	This paper
Exact Majority	Algorithm	$O(\log^3 n)$	$O(\log^2 n)$	This paper
Leader Election	Algorithm	$O(\log^{5.3} n \log \log n)$	$O(\log^2 n)$	This paper

Figure 1: Summary of results and relation to previous work.

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Lower Bound (idea)

- Two-Step argument
- First: Hypothetical algorithm converges faster than allowed by the lower bound, set of low count states can be “erased”
- Second: Engineer examples to contradict the correctness of that algorithm
- Leader Election: Remaining low count states are set of all potential leaders
- Majority: Remaining low count states could sway the outcome of majority

Lower Bound (Leader)

COROLLARY 3.1. *Any monotonic population protocol with $|\Lambda_n| \leq 1/2 \log\log n$ states for all sufficiently large number of agents n that stably elects at least one and at most $\ell(n)$ leaders, must take $\Omega\left(\frac{n}{144^{|\Lambda_n|} \cdot |\Lambda_n|^6 \cdot \ell(n)^2}\right)$ expected parallel time to convergence.*

- Monotonic: Number of states cannot decrease with increasing node count

Lower Bound (Majority)

COROLLARY 3.2. *Any monotonic population protocol with $|\Lambda_n| \leq 1/2 \log\log n$ states for all sufficiently large number of agents n that stably computes correct majority decision for initial configurations with majority advantage ϵn, must take $\Omega\left(\frac{n}{36^{|\Lambda_n|} \cdot |\Lambda_n|^6 \cdot \max(2^{|\Lambda_n|}, \epsilon n)^2}\right)$ expected parallel time to convergence.*

- Monotonic: Number of states cannot decrease with increasing node count

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Synthetic Coin Flips

- Problem: We need a random coin parameter in our state, but states are deterministic (e.g. no randomness)
- Solution: Synthetic Coin Flips
- When x and y interact, they flip their own values
- E.g. $value' = 1 - value$
- Reminder: Interactions are chosen uniformly at random
- Randomness is extracted from the scheduler
- They prove that w.h.p. the distribution quickly becomes uniform
- Important: Happens independently from the algorithm

Lottery Election

- All nodes start in the same state, comprised of parameters:
  - coin = {0, 1} (initially 0)
  - mode = {seeding, lottery, tournament, minion} (initially seeding)
  - payoff, level, counter, phase, ones
- 4 Modes:
  - Seeding Mode, Lottery Mode, Tournament Mode, Minion Mode
- Fix a parameter $m \geq (10 \log n)^2$
- Protocol will use $O(m)$ states per node

Lottery Election – Seeding Mode

- Used to mix the coin parameter close to uniform random
- $payoff, level = 0$
- $counter = 4$
- In the first four interactions, decrease counter (and flip $coin$)
- When counter reaches 0, move on to Lottery Mode

Lottery Election – Lottery Mode

- Used to generate payoff values
- Higher values are less likely, finding a leader becomes easier
- Increment $payoff$ when partner has $coin = 1$
- When partner has $coin = 0$ or $payoff = \sqrt{m}$, move on to Tournament Mode

Lottery Election – Tournament Mode

- Forces agents to compete
- Generates additional tie-braking random values (level)
- Initialize $level = 0$
- $level$ is incremented if agent consecutively sees $\Theta(\log payoff)$ coins set to 1
- This is implemented using $phase$ and $ones$
- Level is capped at $\frac{\sqrt{m}}{\log m}$

Lottery Election – Tournament Mode

- When 2 agents x and y meet, compare:
  - $x.payoff$ and $y.payoff$
  - If payoff equal: compare $x.level$ and $y.level$
  - If level equal: compare $x.coin$ and $y.coin$
- The smaller valued agent goes into $minion\ mode$
- It adopts $level$ and $payoff$ of opponent

Lottery Election - Minion Mode

- Keeps record of the maximum $(payoff, level)$ pair ever seen
- Propagates leaders with high $payoff$
- Helps eliminate other contenders
- Important: $coin$ value not used as tie-breaker

Lottery Election - Complexity

- $coin, counter, modes, ones$ are in $O(1)$
- $payoff$ is limited to $O(\sqrt{m})$
- $level$ is limited to $O\left(\frac{\sqrt{m}}{\log m}\right)$
- $phase$ is limited to $O(\log m)$ since:
  - $\Theta(\log payoff) = \Theta(\log \sqrt{m}) = \Theta\left(\frac{1}{2}\log m\right) = \Theta(\log m)$
- State size: $O(1) \times O(\sqrt{m}) \times O\left(\frac{\sqrt{m}}{\log m}\right) \times O(\log m) = O(m)$
- m was set to $(10\log n)^2$ so state size is $O(log^2 n)$
- They also prove that it takes $O(log^{5.3} n \log\log n)$ parallel time

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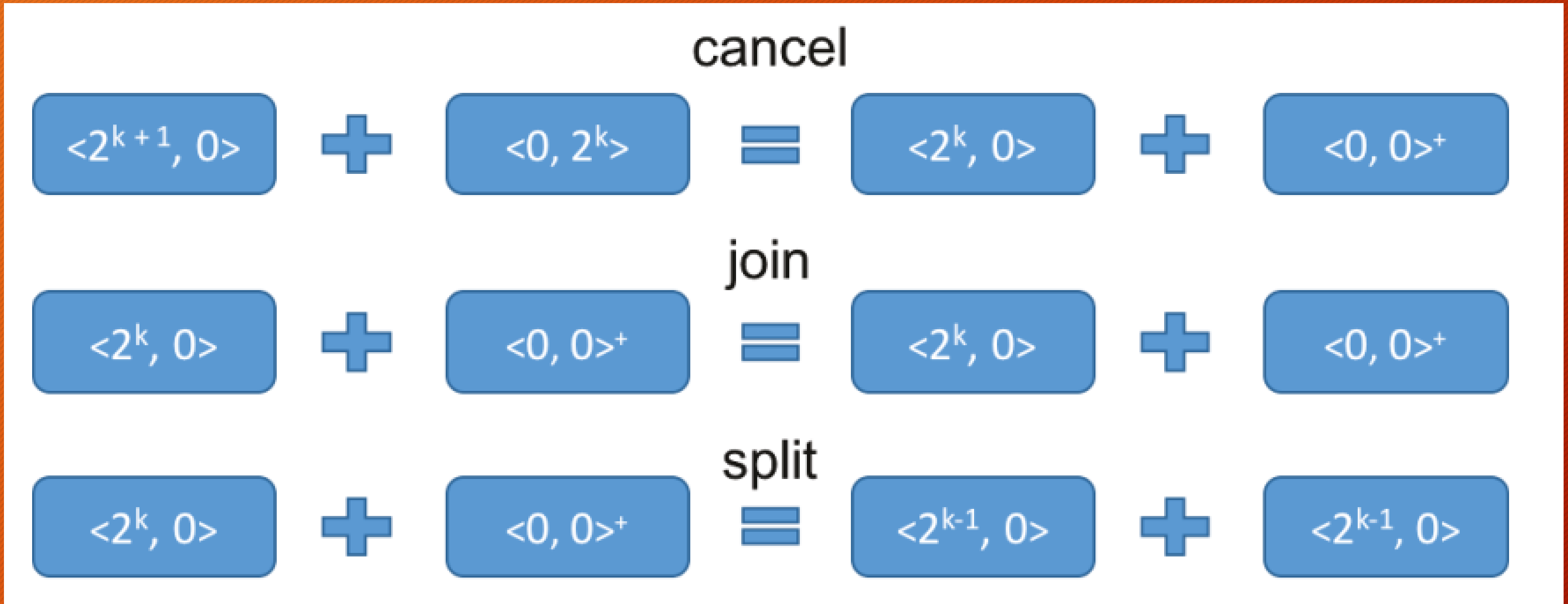
Split-Join Majority

- State: A is positive, B is negative
- To limit state space:
  - State is $\langle x, y \rangle$ where $x, y \in \{0, 1, 2, 2^2, \dots, 2^{\lceil \log n \rceil}\}$
  - $value(\langle x, y \rangle) = x - y$, so x is "positive" and y is "negative"
- Opinion A starts as $\langle 2^{\lceil \log n \rceil}, 0 \rangle$
- Opinion B starts as $\langle 0, 2^{\lceil \log n \rceil} \rangle$
- Strong states: non-zero
- Weak states: $\langle 0,0 \rangle^+$ or $\langle 0,0 \rangle^-$

Split-Join Majority – Transition Rules

- When two agents interact, operations cancel, join, split are carried out



Split-Join Majority – Correctness/Complexity

- Since all operations preserve the sum
- And the initial sum is leaning to one side
- It is impossible for all agents to sway to the “wrong” side
- The Authors prove that the algorithm is guaranteed to converge in $O(log^3 n)$ parallel time in expectation and w.h.p.

Questions?