Quantum graph algorithms

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Basics

Quantum states

- A bit is 0 or 1, a qubit is in a superposition of $|0\rangle$ and $|1\rangle$:

$$|\psi\rangle = \alpha_0 |0\rangle + \alpha_1 |1\rangle$$

- If we measure then we get one outcome.
  The probability of measuring $|0\rangle$ is $|\alpha_0|^2$.
  The probability of measuring $|1\rangle$ is $|\alpha_1|^2$.
- Quantum states are normalized complex vectors, the classical states $|0\rangle, |1\rangle, |2\rangle, \ldots$ form a basis.
- For a qubit:

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

- We combine qubits to create bigger states via tensor products.

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- H changes $|0\rangle$ and $|1\rangle$ into $\frac{|0\rangle+|1\rangle}{\sqrt{2}}$ and $\frac{|0\rangle-|1\rangle}{\sqrt{2}}$.

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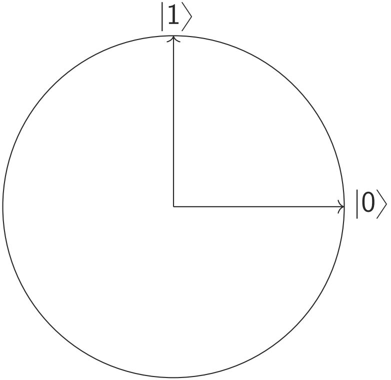

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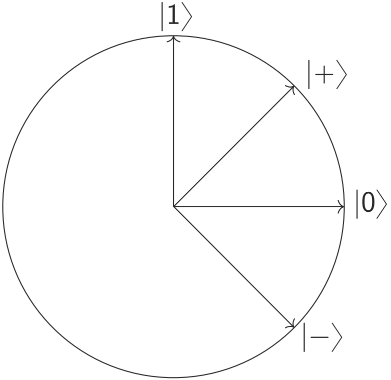

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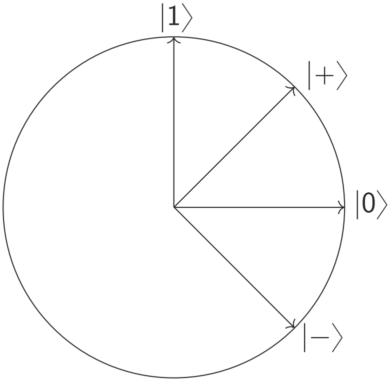
This is a X gate! Z is just X in the $\{|+\rangle, |-\rangle\}$ basis (and vice versa).

Reflections

We can also see this in our image.



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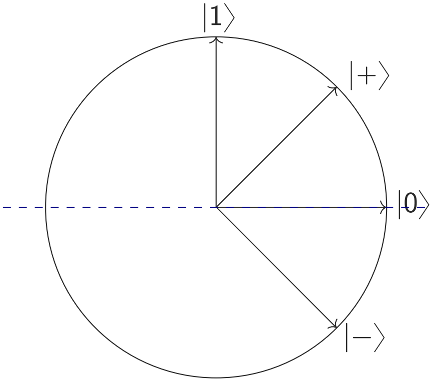

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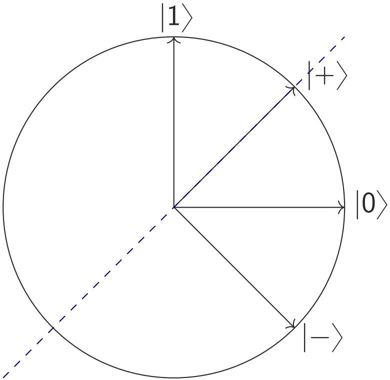

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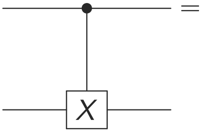

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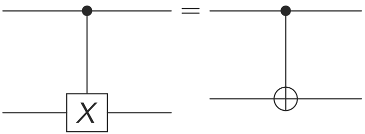

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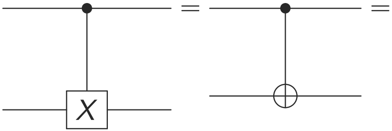

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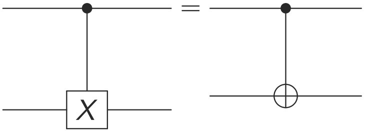

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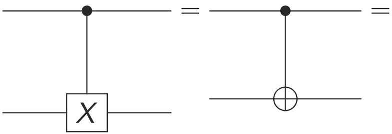

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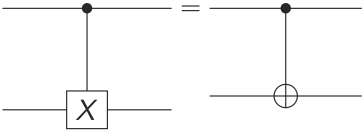

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- Unitaries always have an inverse
  $\Rightarrow$ quantum circuits are always reversible.

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- Is we just measure then the success probability is $p = |\alpha_G|^2$.

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- Everything is in a 2-dimensional subspace.

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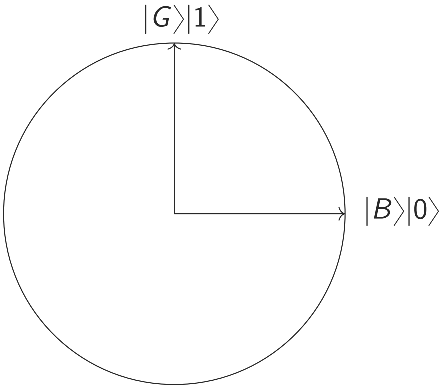

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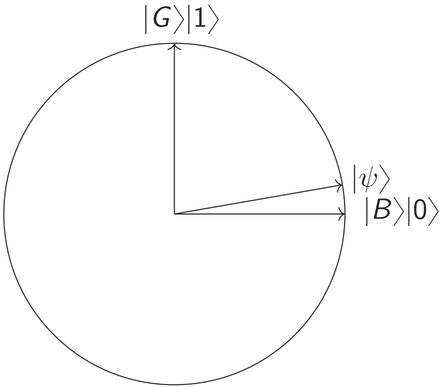

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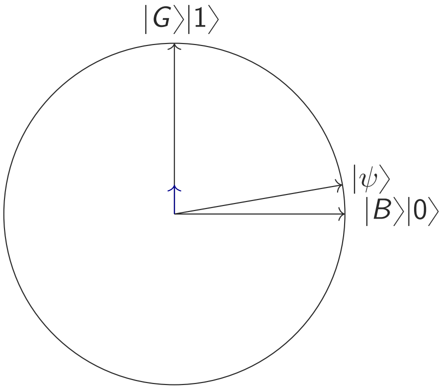

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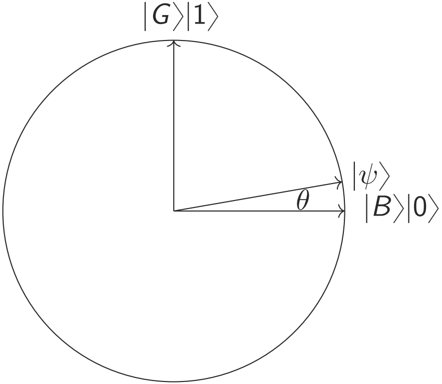

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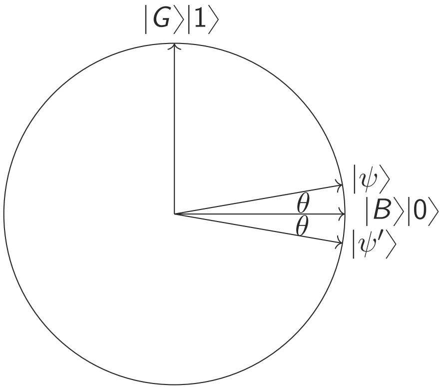

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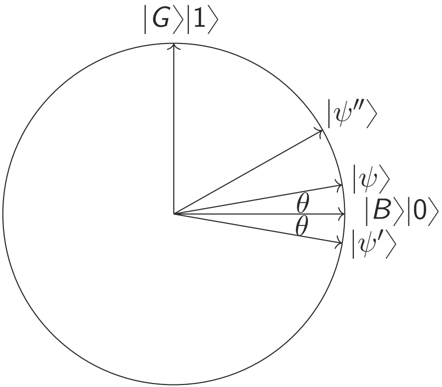

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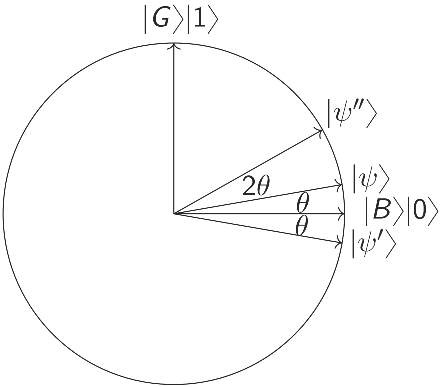

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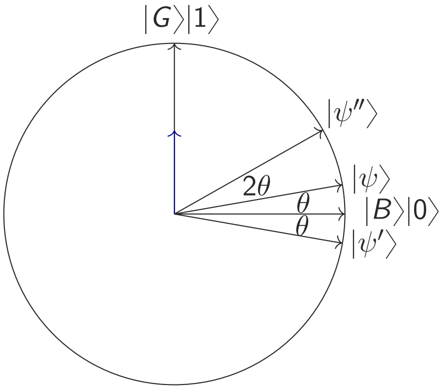

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- Since $\theta = \arcsin\alpha_G$ we want

$$k \approx \left(\frac{\pi}{2\arcsin\alpha_G} - 1\right)/2$$

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- The product is a <u>rotation</u> $\mathcal{A}$, with angle 2θ.
- After k iterations of $\mathcal{A}$ we get

$$\sin\left((2k+1)\theta\right)|G\rangle|1\rangle + \cos\left((2k+1)\right)|B\rangle|0\rangle$$

- If $(2k+1)\theta \approx \pi/2$ then $\left|\sin\left((2k+1)\theta\right)\right|^2 \approx 1$.
- Since $\theta = \arcsin \alpha_G$ we want

$$k \approx \left(\frac{\pi}{2\arcsin \alpha_G} - 1\right)/2 = \mathcal{O}\left(\frac{1}{|\alpha_G|}\right)$$

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- Nice, but can we actually implement these reflections?

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Use that $|\psi\rangle = U\,|0\rangle$:

1. Apply U^{-1} to map $|\psi\rangle$ to $|0\rangle$.
2. Reflect through $|0\rangle$.
3. Apply U to map $|0\rangle$ to back to $|\psi\rangle$.

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To find all: $O(\sqrt{Nk})$

Graphs

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$$\sum_{c=2}^{n} n\sqrt{1/(c-1)} \leqslant n \int_0^n c^{-1/2} dc = O(n^{1.5})$$

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Application: Matching in $O(V\sqrt{E})$

That was it!