More on co-nondeterministic algorithms

Maximum flow problem

Given directed graph with positive edge weights ("capacities"), source s, sink t

An s-t flow is a mapping $f: E \to \mathbb{R}^+$ s.t.

(1) $f(u,v) \leq c(u,v) \quad \forall (u,v) \in E$ (capacity constraint)

(2) $\underbrace{\sum_{u:(u,v)\in E} f(u,v)}_{\text{incoming flow}} = \underbrace{\sum_{u:(v,u)\in E} f(v,u)}_{\text{outgoing flow}} \quad \forall v \in V \setminus \{s,t\}$ (flow conservation)

The value of a flow f is defined as $\sum_{v:(s,v)\in E} f(s,v)$.

Maximum flow problem: find flow f ~~or~~ of maximum value

Decision version: is there a flow of value $\geq t$?

State of the art (essentially):
- $O(mn)$ [King, Rao, Tarjan '94] [Orlin '13]
- $O(m\sqrt{n} \log U \operatorname{polylog}(n))$ [Lee, Sidford '14]

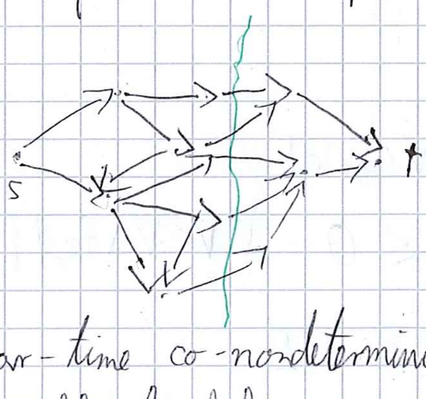
An s-t cut is a partition of V into two sets $S \ni s$, $T \ni t$

The value of a cut (S,T) is $\sum_{\substack{(u,v)\in E \\ u \in S,\, t \in T}} c(u,v)$

Theorem [Ford / Fulkerson '56, Elias et al. '56]

~~value of max s-t cut = valu~~

value of max. s-t flow = value of min. s-t cut



(*) Consequence: (interesting)

No reduction from OV to max flow under NSETH

Linear-time co-nondeterministic algorithm ($\exists$ flow of value $\geq t$?)

- Guess flow f, check capacity constraints and flow conservation, check value $\geq t$
- Guess cut (S,T) (guess for every node if it's in S or T), compute value of cut and check that it is $< t$

(*)

Linear programming

maximize $c^T x$ $\quad$ $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$, $A \in \mathbb{R}^{m \times n}$ given

subject to $Ax \leq b$ $\quad$ $x \in \mathbb{R}^n$ unknown

and $x \geq 0$ $\quad$ A and b represent m constraints

dual problem:

minimize $b^T y$

subject to $A^T y \geq c$

and $y \geq 0$

Theorem (LP Duality) [Tucker '48]

- For all feasible primal solutions x and all feasible dual solutions y (i.e. s.t. $x \geq 0$, $Ax \leq b$, $y \geq 0$, $A^T y \geq c$):
  $$c^T x \leq y^T A x \leq y^T b$$
- For optimal primal and dual solutions x^* and y^*:
  $$c^T x^* = b^T y^*$$

Remark: LP duality gives a trivial co-nondeterministic algorithm with running time depending on structure of A

Linear program for max flow:

One variable $f(u,v)$ per edge $(u,v) \in E$

maximize $\sum_{(u,v) \in E} f(u,v)$

subject to
$$f(u,v) \leq c(u,v) \quad \forall (u,v) \in E$$
$$\sum_{u:(u,v) \in E} f(u,v) - \sum_{u:(v,u) \in E} f(v,u) \leq 0 \quad \forall v \in V \setminus \{s,t\}$$
$$f(u,v) \geq 0 \quad \forall (u,v) \in E$$

Remark: One can show that the max-flow min-cut theorem follows ~~fr~~ from LP duality