A Nonlinear Unitary Framework for Quantum State Reduction: a phenomenological approach

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Si nemo ex me quaerat, scio; si quaerenti explicare velim, nescio.
(Augustine on the nature of time)

Abstract. *A nonlinear self-adjoint operator on a two-dimensional Hilbert space is constructed to force a quantum state, composed of two orthogonal states through linear superposition, into one of these two alternatives (reduction). Associated with this is a bifurcation-process of a collapsing state-vector at the site of two detectors: since there are no discernible experiments for nonunitary schemes of quantum state reduction, this paper provides an implicit nonlinear dynamics of a collapse on a certain time interval, without rejecting unitarity. Set against a large number of explicit reduction-proposals in the literature, the present framework should be viewed as a phenomenological nonlinear unitary scheme that might simulate randomness in quantum physics; hence the notion of quantum-probability appears less inaccessible. So, this essay addresses essential problems of epistemology in quantum physics .*

1. Introduction

The fundamental question to be asked is whether *quantum state reduction* ("collapse of the wave-packet") is something mental (in the eye of the beholder, as it were), or some *real process in time* (independent of the observer‘s presence). The Schrödinger equation does not settle this question: the "reduction mechanism" is not within the realm of orthodox quantum dynamics [1] which admits a *linear unitary evolution* of some prescribed quantum state, up to the point of measurement.

Philosophically speaking, most physicists would consider any dynamical description of the collapse of the wave-packet as being rather superfluous: they claim that the wave function has no *objective* meaning (with respect to *physical reality*), e.g. a collapse-model would not improve on the predictive power of the standard theory. The present author is not convinced that the philosophy of quantum mechanics should solely be based on algorithms for merely computing data relating to specific experiments. As will be pointed out in the discussion, the actual *timing* of the collapse is of epistemological significance, if reduction is *physically real*. One of the objectives of this paper then is to challenge the almost *dogmatic* view that any unitary evolution of the wave function should necessarily be linear.

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Now, the basic idea behind the proposed framework is easily put into the context of major issues in quantum theory: might perhaps the correct probabilistic interpretation of quantum mechanics come about by viewing quantum state reduction as a nonlinear unitary bifurcation-process which simulates randomness ?

In the following an instructive model of a nonlinear extension of the Schrödinger equation is presented, so as to provide a simple framework for a collapse. This nonlinear model, yet unitary, is *phenomenological* as opposed to other schemes that make explicit dynamical assumptions (compare with [2],[3] and the references therein), though some of these exhibit rather arbitrary features and are probably not checkable; e.g. in this essay there is no direct appeal to *entanglement* with the outside world which, as in the case of *environmental* proposals [4], would not only have to include explicit details of detectors but ultimately also the observer‘s mental processes! Finally, since this paper presents a mere *phenomenological* framework for a physical process, it is based on minimal assumptions.

To this end, consider a photon beam incident on a semi-transparent (“half-silvered”) mirror (or possibly some different arrangement of Stern/Gerlach-type [1]): this beam splits up into a reflected ray 1 and a second ray 2 that passes through the mirror. In Dirac‘s notation we have for the quantum state of the photon: $|p\rangle = \frac{1}{\sqrt{2}}\,(|1\rangle + |2\rangle)$, at the *time-point* $t = 0$, say (for the sake of simplicity, no complex phase factors are considered); the *vectors* $|1\rangle$ and $|2\rangle$ being orthonormal.

When two detectors D_1 and D_2, to be associated with the split beam, are suitably placed on either side of the mirror (not necessarily equidistant [2]) , the interpretation of quantum mechanics tells us that in this case the *quantum-probability* for detecting the photon is given by the *squared amplitudes* $|\langle 1|p\rangle|^2 = |\langle 2|p\rangle|^2 = \frac{1}{2}$ (time-independent). Upon detection of the photon, reduction then means that the vector $|p\rangle$ is *forced* either into the *state* $|1\rangle$ at the site of D_1 , or into the *state* $|2\rangle$ at the site of D_2.

2. A nonlinear unitary framework for reduction

Let us suppose that the linear Schrödinger equation holds for $t \in (-\infty, 0)$. Then at $t = 0$, the detectors start to *interact* with the split beam, this *interaction* lasting for τ seconds. The photon state $|p\rangle$ undergoes “rotation” in a two-dimensional Hilbert space (spanned by $|1\rangle$ and $|2\rangle$), s.t. the reduction is completed at $t = \tau$. In order to describe this process, in a *coordinate-free* representation, we simply write:

$$|p\rangle = \exp(-i\int_0^t \mathcal{R}\,dt)\left[\frac{1}{\sqrt{2}}(|1\rangle + |2\rangle)\right], \quad \forall t \in [0, \tau],$$

s.t. the vector $|p\rangle$ is reduced either to $|1\rangle$ or $|2\rangle$ at $t = \tau$; here $\mathcal{R}$ is an appropriate self-adjoint operator (preserving the length of $|p\rangle$), and $[0, \tau]$ is the *collapsing time-interval*.

The actual crux is the *time-parameter* τ, indicating that $\mathcal{R}$ is to be associated with a *real* process *induced* by these detectors. Notice that the evolution is still unitary, but $\mathcal{R}$ being now a nonlinear operator (depending on the amplitudes). This is probably an unconventional approach, since here the concept of unitarity is applied to a nonlinear evolution; yet, within the scope of the present paper, this is the obvious way of incorporating *state-vector reduction* into a generalized scheme of a *unitary quantum dynamics*. Moreover, as pointed out in the appendix, this approach might have some

[1] There are now various international work groups engaged in testing orthodox quantum theory, mainly in the field of quantum optics: `http://www.physics.mq.edu.au/~drice/quoptics.html`

[2] Careful experimental checks are still needed (A. Zeilinger, priv. com.).

significant bearing on the notion of *randomness* in quantum physics!

Instead of Schrödinger‘s linear equation we now have a nonlinear evolution equation:

$$i\frac{d\,|p\rangle}{dt} = \mathcal{R}\,|p\rangle\;,\; t \in (0,\tau),$$

where $\mathcal{R}$ depends on the amplitudes of the state $|p\rangle$.

2.1. The meaning of the nonlinear *reduction operator* $\mathcal{R}$

We choose the following ansatz for the state vector ($\forall t \in [0,\tau]$) :

$$|p\rangle = |1\rangle\,(\cos\phi) + |2\rangle\,(\sin\phi)\;,\;\phi \in \left[0, \frac{\pi}{2}\right],$$

operating (for the sake of simplicity) within the first quadrant, with ϕ depending on t; (the initial value $|p\rangle\,|_{t=0}$ comes from the solution of the linear Schrödinger equation). Then

$$\frac{d\,|p\rangle}{dt} = -\left[\langle 2|p\rangle\,|1\rangle - \langle 1|p\rangle\,|2\rangle\right]\,\frac{d\phi}{dt}\;;$$

thus by rearranging, and observing the evolution equation:

$$\mathcal{R} = -i\,(|1\rangle\,\langle 2| - |2\rangle\,\langle 1|)\;\omega,\;(\omega = \frac{d\phi}{dt})$$

where ω is the “angular speed” at which the reduction of the state vector takes place. [3]

Diagonalization: By introducing a unitary transformation $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}$, note that $\frac{1}{\sqrt{2}}(|1\rangle \pm i\,|2\rangle)$ are the normalized eigenvectors belonging to the eigenvalues $\mp\omega$.

2.2. A choice for the *angular speed* ω

There is no clue from first principles to start with. A simple *phenomenological* model would be the following nonlinear deterministic equation:

$$\frac{d\phi}{dt} = -\frac{1}{2\,\tau\cos 2\phi},\; t \in (0,\tau)$$

with $\phi(0) = \frac{\pi}{4}$. Thus

$$\mathcal{R} = \frac{i\,(|1\rangle\,\langle 2| - |2\rangle\,\langle 1|)}{2\,\tau\;\left(|\langle 1|p\rangle|^2 - |\langle 2|p\rangle|^2\right)},$$

depending nonlinearly on the amplitudes (now *time-dependent*).

Finally, integration yields:

$$\sin 2\phi = \frac{\tau - t}{\tau},\;\forall t \in [0,\tau].$$

[3] The matrix representation of $\mathcal{R}$ is then $\omega\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, where ω depends on the amplitudes (see below)!

Hence the state vector $|p\rangle$ is indeed reduced, either to $|1\rangle$ or $|2\rangle$ at $t=\tau$, according to the solution set: $\left\{\phi=0, \phi=\frac{\pi}{2}\right\}$; we may thus say that the *initial angle* $\phi(0)$ is a *bifurcation-point*.

In a more general setting a beam splitter is not exactly "half-silvered", so at $t=0$:

$$|p\rangle = |1\rangle\left(\cos\phi_0\right) + |2\rangle\left(\sin\phi_0\right),\ 0<\phi_0<\frac{\pi}{2}.$$ [4]

For example, the equation for the *angular speed* is generalized to:

$$\frac{d\phi}{dt} = -\frac{\tan\,2\phi}{2(\tau-t)},\ t\in(0,\tau),$$

with $\phi(0)=\phi_0$; thus:

$$\sin 2\phi = \left(\frac{\tau-t}{\tau}\right)\sin 2\phi_0, \quad \forall t\in[0,\tau]\,.$$

Note that there is no analogue in classical mechanics: hence, as to the construction of $\mathcal{R}$, there is no *correspondence principle* that would hint at an explanation for a collapse: so, in the following we do not speculate on an explicit dynamics for detectors that would bring about the reduction of a state-vector; by setting up a mere *phenomenological* framework for a collapse, we may simply claim that detectors of this kind certainly exist!

Actually, we do not need an explicit knowledge of $\frac{d\phi}{dt}$: Define, for the sake of simplicity, $\xi = \cos\phi,\ \eta=\sin\phi$, then:

$$\frac{d\xi\eta}{dt} = \left(\xi^2-\eta^2\right)\frac{d\phi}{dt},\ \text{s.t. } \xi\eta = -\int_t^\tau \left(\xi^2-\eta^2\right)d\phi(t);$$

the reduction is completed at $t=\tau$, i.e. $\xi\eta=0$, with ϕ either 0 or $\frac{\pi}{2}$. A general formula is then given by:

$$\frac{d\phi}{dt} = -\frac{g(\xi,\eta,t)}{\xi^2-\eta^2},\ \text{s.t.}\ \int_0^\tau g(\xi,\eta,t)\,dt = \xi(0)\eta(0),$$

with g being some appropriate function.

Generalization to complex amplitudes (accounting for interference-effects): Let $g(\xi,\eta,t)$ be an appropriate complex valued function, integrable on $[0,\tau]$; a unitary evolution equation for reduction on complex numbers then reads:

$$i\frac{d\,|p\rangle}{dt} = -\frac{1}{|\xi|^2-|\eta|^2}\begin{pmatrix} 0 & -ig \\ i\overline{g} & 0 \end{pmatrix}|p\rangle\,,\ t\in(0,\tau),\ \text{with}\ \int_0^\tau g(\xi\eta,t)\,dt = \xi(0)\overline{\eta(0)};$$

phenomenologically speaking, this *condition* on $[0,\tau]$ means that a collapse certainly occurs!

Incidentally, this approach is not restricted to a 2 dimensional Hilbert space: one needs at most $N(N-1)/2$ appropriate g-functions, equal to the number of independent entries in the matrix representation of $\mathcal{R}$, with $|p\rangle\in l^2(N),\ N\geq 2$. To be sure, it is because of these functions that detectors

[4]The associated quantum probabilities are given by $|\cos\phi_0|^2$ and $|\sin\phi_0|^2$, resp.: At this stage, since the present approach is strictly deterministic (exploiting merely nonlinear instability), these probabilities are left unexplained. However, notice that *reduction* to a single quantum state follows from the validity of the probability-interpretation of the theory; the *epistemological* point is whether this reduction is a mere *subjective fact* (depending on the presence of an observer) or an *objective part* of the physical world, i.e., a *process* that evolves in *time* (see appendix)!

select a single outcome from a set of superposed states $[\xi_1, \ldots, \xi_N]$, with $\sum_N |\xi_i|^2 = 1$; therefore, without explicitly specifying the measurement process, these functions *model* a dynamical link between suitable detectors and the observed system; it then appears, for example, that *random* photon-counts are not simply a consequence of certain *environmental decoherence*-effects (also see remark at the end of the appendix), as might be suggested in [4]!

Remark 1: The eigenvectors of the diogonalized *reduction operator*, call it $\widetilde{\mathcal{R}}$, with matrix representation $\omega \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, follow a Schrödinger-type evolution:

$$i\frac{d\,|r\rangle}{dt} = \widetilde{\mathcal{R}}\,|r\rangle\,,\ t \in (0,\tau),\ |r\rangle\,|_{t=0} = \frac{1}{\sqrt{2}}(|1\rangle \pm i\,|2\rangle);$$

hence

$$|r\rangle = |r\rangle\,|_{t=0}\, e^{\pm i(\phi-\phi_0)},$$

where the *rotation* is either clockwise or counter-clockwise.

Remark 2: Let the reduction be a “mental act” (*Feststellung eines Zustandes* [1]); then the result of an observation is simply given by the *quadratic equation*:

$$\langle p|p\rangle - (\langle p|2\rangle + \langle 1|p\rangle) = 0,\ \text{with}\ |p\rangle\ \text{either}\ |1\rangle\ \text{or}\ |2\rangle\,;$$

our initial knowledge of the situation (the split beam) has changed *discontinuously*. In case of entanglement, involving two or more correlated quantum objects $p', p'', p''', \ldots$ in this interpretation one replaces each base vector with products $|'\rangle\,|''\rangle\,|'''\rangle \ldots$., s.t. upon reduction the overall $|p\rangle$ is either $|1'\rangle\,|1''\rangle\,|1'''\rangle \ldots$, or $|2'\rangle\,|2''\rangle\,|2'''\rangle \ldots$. This is the so-called *reduction postulate* which, from a mere *pragmatic* point of view, is certainly not wrong; but, in the light of the present paper, does this postulate offer a *complete* description?

Concluding Remark: By making use of a rather simple bifurcation scheme, this paper expresses a unifying aspect within different areas of quantum mechanics through a unitary Schrödinger-type evolution equation :

$$i\frac{d\,|p\rangle}{dt} = \mathcal{Q}\,|p\rangle\,,\ (t \in \mathbb{R}^1);$$

here this self-adjoint operator $\mathcal{Q}$ is either $\mathcal{H}$ (the linear *Schrödinger-Hamiltonian*), or $\mathcal{R}$ (the nonlinear *reduction operator*), according to whether $t \in (-\infty, 0)$, or $t \in (0,\tau)$, say; the linear Schrödinger equation only holds up to the *time* of measurement which sets about at $t = 0$, whereas the nonlinear evolution equation for reduction relates to the actual *measurement process* which is completed at $t = \tau$.

Note that we are dealing with different *elements of reality*: there are photons at the quantum-level, and secondly there are *macroscopic gadgets* (beam splitters and detectors). So, I believe that τ depends on the *admissible size* of these devices, and possibly on the frequency ν of the beam (being an eigenvalue of $\mathcal{H}$), s.t. $\nu < \nu_0$; here ν_0 should be some *cutoff* for the *coherent splitting* into rays of comparable amplitudes, i.e.: for frequencies ν much larger than ν_0, I would not expect that quantum-mechanical interference effects (as in interferometry) can be resolved by any kind of experiment. Also note that there are various views on the *dividing line* between *micro*- and *macro* objects, as discussed in [5] and very recently in [10].

3. Discussion

The view I take here is still somewhat formal, but perhaps soothing to those who believe that in orthodox quantum theory there is something amiss, i.e. the so-called “measurement process” is not part of the general scheme. It is not intended to consider explicit reduction-proposals, so as to unfold the *true nature* of the collapse (even considering *superpositions of gravitational fields* [2],[3]), nor to compete with explicit stochastic and nonunitary investigations [4],[5],[6]: by introducing a suitable framework for *state-vector reduction* without departure from unitarity, there is no need to go beyond the rules of standard quantum theory ; phenomenologically speaking, what really matters is the partitioning of the *timeline*, i.e. the interval $(-\infty, \tau)$: the state-vector evolves linearly on the *undisturbed interval* $(-\infty, 0]$, and collapses nonlinearly on the *measuring interval* $[0, \tau]$!

This phenomenological model represents an implicit nonlinear dynamics for the *reduction process* with a critical *time-parameter* τ; it hints at an essential *time-scale* for the collapse of superposed quantum states *interacting* with a *macroscopic* measuring apparatus. One might conjecture that τ relates to the *admissible size* of any measuring device, assuring *macroscopic amplification* of single quantum events. However, at present it would be difficult to ascertain a definite empirical basis for τ; so, experimenters are urged to pursue this issue further : for a better insight into the epistemology of quantum physics rather intricate measurements, e.g. the actual *timing* of the collapse on $[0, \tau]$, have yet to be performed!

The issue in question is not a matter of numerically estimating τ which, as in the case of explicit modifications of quantum mechanics, depends rather strongly on specific modelling assumptions; this is a matter of principle with far-reaching *epistemological* consequences [7]: in case that τ is not determinable by measurement (irrespective of the underlying dynamics), the proponents of *objective quantum state reduction* are likely to be led back to the “Copenhagen interpretation” (for a *modern version* see [8]).

Appendix

Let us consider a *probabilistic game* for two players, without explicitly knowing its ingredients: it is, at any rate, a game involving probabilities P_1 and P_2, referring to the likelihood of winning, s.t. $P_1 + P_2 = 1$: such a game may have an expected *time*-lapse of τ seconds. Furthermore, assume that an appropriate function $g(\xi, \eta, t)$ exists, with $\int_t^\tau g\, dt = \xi\eta$; for the sake of simplicity, here (ξ, η) are real valued *game-variables*, depending on $t \in [0, \tau]$, s.t. $|\xi|^2 + |\eta|^2 = 1$. The question arises whether there exists a game, s.t. $P_1 = |\xi(0)|^2$, $P_2 = |\eta(0)|^2$; for example, the first player wins the game when $|\xi| = 1$, $\eta = 0$ at $t = \tau$. (Note, in the main text the *quantum-probabilities* in question are determined by the absolute square of the amplitudes at the onset of *measurement*).

Probabilities are usually assigned to *random* processes [9]. In the present context the element of *randomness* might be due to bifurcation, the outcome ($|p\rangle$ either $|1\rangle$ or $|2\rangle$) is not computable; for example, in a Mach-Zehnder interferometer there is no certainty as to which path a photon will take after entering the apparatus. On a macroscopic scale, however, when one tosses a *fair* coin, it is assumed that the probability for either outcome (heads or tails) is 50%; then according to Newtonian mechanics, if we would know all the physical details at some given *time-point*, we could actually compute the outcome with certainty, i.e. *classical probabilities* can be *embedded* into classical physics!

However, no such computation for mutually exclusive outcomes is logically possible in quantum theory: principally, in quantum theory there is a lack of *additional information* - as is the case when we try to compute a *unique* number ϕ from a transcendental equation, like $\sin 2\phi = \Omega(t),\ (0 \leq t \leq \tau)$, at $t = \tau$, with $\Omega(\tau) = 0$; in fact, there is no such unique number at $t = \tau$, even though this might be a solution of a *deterministic evolution* equation! So, it appears that within the context of state-vector reduction, the concept of *quantum-probability* is ultimately related to that of *bifurcation* in the present sense.

Remark: For example, consider the *environmental density matrix-approach* without state-vector reduction [4]: it lacks a *genuine probabilistic* interpretation (to be based on a random process), since *randomness* is absent from that theory, as pointed out in [9]. Thus this approach does not solve the *random measurement* problem!

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