```
  if(r<s)
    return 0;
  if(r == s)
    return 1;
  for(count=0;count<s;count++)
  {
    denom = denom*(r-count);
    nom = nom*(count+1);
  }
  return (long)(denom/nom);
}
```

References

[BH95] K. Balakrishnan and V. Honavar. Evolutionary design of neural architectures. Technical Report CS TR #95-01, Iowa State University, January 1995.

[Cha77] Gary Chartrand. *Graphs as Mathematical Models.* Prindle, Weber & Schmidt, 1977.

6 Appendix C

```
#include<stdio.h>

#define NR_GRAPHS 8

long graphs(long,long);

long m,n;

void
main()
{
  for(m=1;m<NR_GRAPHS;m++)
    for(n=1;n<NR_GRAPHS;n++)
    {
      printf("%6d &",graphs(m,n));
    }
}
/*****************************************************/
long
graphs(long m,long n)
{

  long binom(long,long);

  long temp;
  long i,j;

  if((m== 0) || (n==0))
    return 0;
  if((m== 1) || (n==1))
    return 1;
  if(m < n)
  {
    temp = n;
    n = m;
    m = temp;
  }
  temp = 0;
  for(i=1;i<=m;i++)
  {
    for(j=0;j<=i;j++)
    {
      temp = temp + binom(m-j,i-j)*binom(m,j)*graphs(m-j,n-1);
    }
  }
  return temp;
}
/*****************************************************/
long
binom(long r,long s)
{
  long count,
       nom=1, /* Nominator */
       denom=1; /* Denominator */

  if((r == 0) && (s == 0))
    return 0;
```

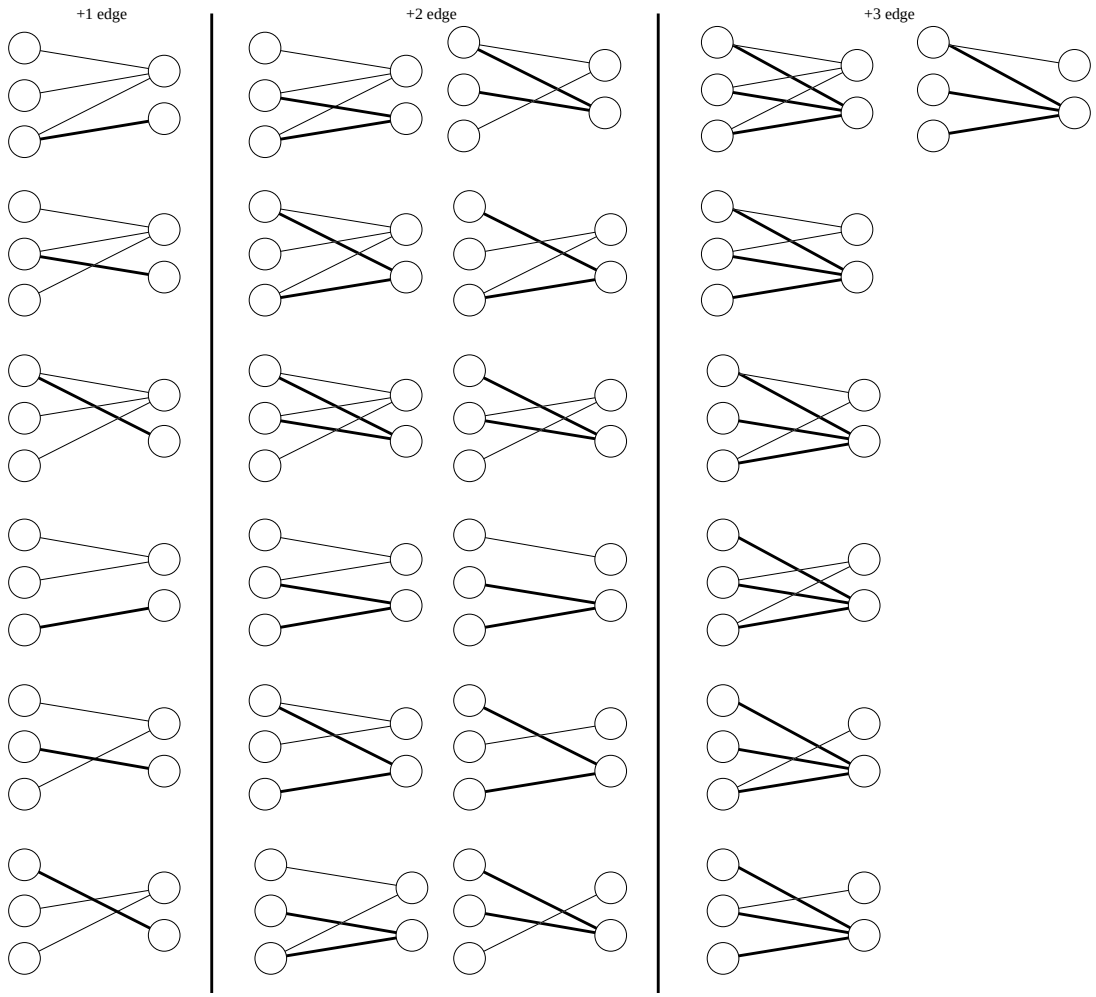

Figure 3: {G(3,2)}

	1	2	3	4	5	6	7
1	1	1	1	1	1	1	1
2	1	7	25	79	241	727	2185
3	1	25	265	2161	16081	115465	816985
4	1	79	2161	41503	693601	10924399	167578336
5	1	241	16081	693601	24997920	831719744	26666532864
6	1	727	115465	10924399	831719744	57366999040	3776451182592
7	1	2185	816985	167578336	26666532864	3776451182592	505874737004544

Table 1: Number of "valid" graphs $g(m, n)$, with $m, n = 1, \ldots, 7$.

Finally, adding up these partial results yields:

$$\left(\begin{array}{c} m \\ 2 \end{array} \right) \left(\begin{array}{c} m \\ 0 \end{array} \right) \mathrm{g(m,n)} + \left(\begin{array}{c} m-1 \\ 1 \end{array} \right) \left(\begin{array}{c} m \\ 1 \end{array} \right) \mathrm{g(m-1,n)} + \left(\begin{array}{c} m-2 \\ 0 \end{array} \right) \left(\begin{array}{c} m \\ 2 \end{array} \right) \mathrm{g(m-2,n)}.$$

$+i$ **new edges** We consider graphs with i edges arriving in vertex r_{n+1}. In this case we can have $0, 1, 2, \cdots, i$ isolated vertices. We proceed as shown in the previous steps and obtain:

$$\left(\begin{array}{c} m \\ i \end{array} \right) \left(\begin{array}{c} m \\ 0 \end{array} \right) \mathrm{g(m,n)} + \left(\begin{array}{c} m-1 \\ i-1 \end{array} \right) \left(\begin{array}{c} m \\ 1 \end{array} \right) \mathrm{g(m-1,n)} + \ldots + \left(\begin{array}{c} m-i \\ i-i \end{array} \right) \left(\begin{array}{c} m \\ i \end{array} \right) \mathrm{g(m-i,n)}.$$

These results can be rewritten as a double sum, where the index of the outer sum runs from 1 to m (the number of edges arriving in r_{n+1}) and the inner sum runs from 0 to i (the number of isolated vertices):

$$\sum_{i=1}^{m} \sum_{j=0}^{i} \left(\begin{array}{c} m-j \\ i-j \end{array} \right) \left(\begin{array}{c} m \\ j \end{array} \right) g(m-j, n).$$

q.e.d.

3 Example for $\{G(3,2)\}$:

In Figure 3 all possible $G(3,2)$ graphs are shown.

4 Appendix A

Theorem 4.1 (First Principle of Mathematical Induction:) *Let m be an integer, and let $S_m, S_{m+1}, S_{m+2}, \ldots$ be statements with the properties:*

1. *S_m is a true statement;*
2. *If S_k is a true statement, then S_{k+1} is a true statement.*

Then each of the statements $S_m, S_{m+1}, S_{m+2}, \ldots$ is true.

The proof is to be found in [Cha77].

Theorem 4.2 (Second Principle of Mathematical Induction:) *Let m be an integer, and let $S_m, S_{m+1}, S_{m+2}, \ldots$ be statements with the properties:*

1. *S_m is a true statement;*
2. *If S_i is a true statement for each $i, m \leq i \leq k$, then S_{k+1} is a true statement.*

Then each of the statements $S_m, S_{m+1}, S_{m+2}, \ldots$ is true.

The proof is to be found in [Cha77].

5 Appendix B

The results of Table 1 were calculated on a Sun SPARC 20. The fields in the table denote the number of valid graphs for m vertices in the set L (columns) and n vertices in the set R (rows).

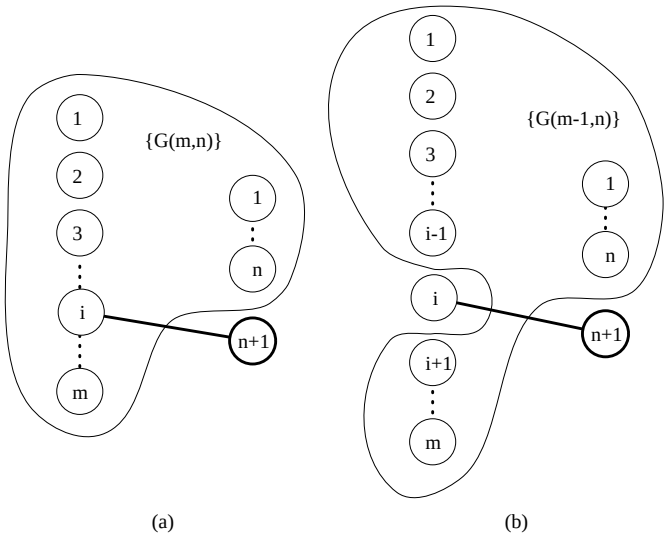

Figure 2: Induction step

graphs.

Now, we consider all cases where the vertex l_{i_1} from which the new edge departs is **isolated**. Thus we count all graphs with one new edge departing from any l_{i_1} where $\deg l_{i_1} = 1$ (see Figure 2.(b)). We therefore have

$$mg(m-1,n) = \begin{pmatrix} m-1 \\ 0 \end{pmatrix} \begin{pmatrix} m \\ 1 \end{pmatrix} g(m-1,n)$$

additional graphs.

+2 new edges: Now, we count all graphs with two new edges arriving in r_{n+1}. We have two vertices l_{i_1}, l_{i_2} where the new edges may depart from. Either none, one, or two of these are isolated vertices. Suppose neither l_{i_1} nor l_{i_2} is an isolated vertex. Obviously, we have $\begin{pmatrix} m \\ 2 \end{pmatrix}$ pairs l_{i_1}, l_{i_2} multiplied by $g(m,n)$. Hence we obtain

$$\begin{pmatrix} m \\ 2 \end{pmatrix} \begin{pmatrix} m \\ 0 \end{pmatrix} g(m,n).$$

In the case of one isolated vertex l_{i_k}, we have $m(m-1) = \begin{pmatrix} m-1 \\ 1 \end{pmatrix} \begin{pmatrix} m \\ 1 \end{pmatrix}$ pairs l_{i_1}, l_{i_2} with one isolated vertex l_{i_k} with $k \in \{1,2\}$. Hence we obtain

$$\begin{pmatrix} m-1 \\ 1 \end{pmatrix} \begin{pmatrix} m \\ 1 \end{pmatrix} g(m-1,n).$$

Consider now two isolated vertices. We have $\begin{pmatrix} m-2 \\ 0 \end{pmatrix} \begin{pmatrix} m \\ 2 \end{pmatrix}$ possible pairs. So for this case, we obtain

$$\begin{pmatrix} m-2 \\ 0 \end{pmatrix} \begin{pmatrix} m \\ 2 \end{pmatrix} g(m-2,n).$$

Theorem 2.1 *The number of all labeled, non-isolated, bipartite graphs $G(m,n)$ is given by*

$$|\{G(m,n)\}| = g(m,n) = \sum_{i=1}^{m}\sum_{j=0}^{i} \left(\begin{array}{c} m-j \\ i-j \end{array} \right) \left(\begin{array}{c} m \\ j \end{array} \right) g(m-j, n-1),$$

with

$$g(r,s) = \begin{cases} 0, & if\ \{r = 0 \vee s = 0\}, \\ 1, & if\ \{\{r = 1 \wedge s \neq 0\} \vee \{s = 1 \wedge r \neq 0\}\}. \end{cases}$$

and

$$g(r,s) = g(s,r).$$

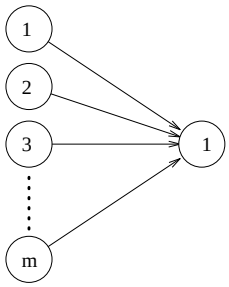

Figure 1: G(m,1) Graph

Proof:

We prove this theorem by induction:

Basis:

$m > 0, n = 0$: In this case $g(m,n)$ is defined to be 0.

$m > 0, n = 1$: In Figure 1 we show the case of one vertex in set R. Obviously there is just one possibility to construct a labeled, non-isolated, bipartite graph.

The next step in the proof covers the induction step from n to $(n+1)$. For all the cases to be covered consider that one vertex r_{n+1} is added to the set R. The essential task is to count the graphs with new edges arriving in r_{n+1}.

Induction Step: $n \rightarrow n+1$

$m > 0, n+1$:

+1 new edge: We consider all graphs with one edge arriving in the new vertex r_{n+1} (see Figure 2.(a)). We have m possibilities where this edge can depart from multiplied by the number of $g(m,n)$ which is by induction well defined. Note that the vertex l_{i_1} where the new edge departs from is a member of all graphs $\in \{G(m,n)\}$, meaning that $\deg l_{i_1} \geq 2$, otherwise it would not have been a member of any $G(m,n)$. This results in

$$mg(m,n) = \left(\begin{array}{c} m \\ 1 \end{array} \right) \left(\begin{array}{c} m \\ 0 \end{array} \right) g(m,n)$$

1 Introduction

Neural Networks are a powerful Artificial Intelligence paradigm for solving various kinds of problems. Facing the huge number of papers published in this field, it seems surprising that there have seldom been efforts to actually count the number of "possible" network topologies. This report presents a first approach to this problem. Of course, in graph theory it has been dealt with counting graphs, to which it comes to, eventually. However, the authors of this report wanted to have graphs with specific properties. The task to find a formula for the number of possible graphs turns out to be harder than it seems to be at first glance.

This report deals with Neural Network topologies containing one input layer with m input neurons and n output neurons in the output layer. In a Neural Network no isolated vertices are allowed, neither in the input, nor in the output layer. Additionally, no intra-layer connections are allowed. In this way, a set of graphs $\{G(m,n)\}$ is formed, as will be defined lateron. Of course, the graphs in this set are bipartite from the viewpoint of graph theory.

Why is this number important? There have been several approaches to evolutionary design of Neural Network topologies using genetic algorithms (see also [BH95]). The formula in this report presents the precise number of possible topologies with only one input and one output layer. Due to the structure of this formula, it is clear that this number is going to increase immensely (see also Table 1) for even small numbers of input and output neurons. Therefore, it presents a strong argument, why it is useful to exploit genetic search techniques for Neural Network topologies, facing such a huge number of possible topologies.

The authors of this report are currently seeking formulas for nets with several hidden layers. Finding formulas poses a lot more problems, because of possible interactions between hidden layers. In addition, these graphs no longer form bipartite graphs.

2 The Formula

Definition 2.1 *A* **graph** *G is an ordered pair (V,E), in which V is a set called the* **vertex set** *and E is a set consisting of subsets of V with two elements. The set E is called the* **edge set**, *and we say that a and b are joined by an edge if $\{a,b\}$ belongs to E.*

Definition 2.2 *A* **bipartite graph** *is a graph in which the vertices are divided into two disjoint subsets (L,R) in such a manner that there are no edges connecting vertices in the same set.*

Definition 2.3 *A* **labeled** *bipartite graph $G(m,n)$ is a bipartite graph with labeled vertices, where $|L| = m$ and $|R|=n$.*

Definition 2.4 *A* **non-isolated** *bipartite graph $G(m,n)$ is a bipartite graph where the degree of every vertex v, $\deg v \geq 1$. The set of all labeled, non-isolated, bipartite graphs with $|L| = m$ and $|R|=n$ is denoted as $\{G(m,n)\}$.*

Definition 2.5 *The two-valued function*

$$g : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$$

indicates the cardinality of $\{G(m,n)\}$.

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On the Number of Labeled, Non-Isolated, Bipartite Graphs

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Abstract

The authors give a recursive formula for the number of labeled, non-isolated, bipartite graphs. This result will serve as a starting point for further research concerning the number of "valid" Feed-Forward Neural Network topologies.