Geometric Modeling
(WS 2020/21)

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URL of course (VO+PS): Base-URL/teaching/geom_mod/geom_mod.html.

Lecture times (VO): Friday 12:55–14:30.
Venue (VO): T03, Computerwissenschaften, Jakob-Haringer Str. 2.

Lecture times (PS): Friday 11:45–12:35.
Venue (PS): T03, Computerwissenschaften, Jakob-Haringer Str. 2.

Note — PS is graded according to continuous-assessment mode!
Electronic Slides and Online Material

In addition to these slides, you are encouraged to consult the WWW home-page of this lecture:

https://www.cosy.sbg.ac.at/~held/teaching/geom_mod/geom_mod.html.

In particular, this WWW page contains up-to-date information on the course, plus links to online notes, slides and (possibly) sample code.
A Few Words of Warning

- I hope that these slides will serve as a practice-minded introduction to various aspects of geometric modeling. I would like to warn you explicitly not to regard these slides as the sole source of information on the topics of my course. It may and will happen that I’ll use the lecture for talking about subtle details that need not be covered in these slides! In particular, the slides won’t contain all sample calculations, proofs of theorems, demonstrations of algorithms, or solutions to problems posed during my lecture. That is, by making these slides available to you I do not intend to encourage you to attend the lecture on an irregular basis.

- See also In Praise of Lectures by T.W. Körner.

- A basic knowledge of calculus, linear algebra, discrete mathematics, and geometric computing, as taught in standard undergraduate CS courses, should suffice to take this course. It is my sincere intention to start at such a hypothetical low level of “typical prior undergrad knowledge”. Still, it is obvious that different educational backgrounds will result in different levels of prior knowledge. Hence, you might realize that you do already know some items covered in this course, while you lack a decent understanding of some other items which I seem to presuppose. In such a case I do expect you to refresh or fill in those missing items on your own!
Acknowledgments

A small portion of these slides is based on notes and slides originally prepared by students — most notably Dominik Kaaser, Kamran Safdar, and Marko Šulejić — on topics related to geometric modeling. I would like to express my thankfulness to all of them for their help. This revision and extension was carried out by myself, and I am responsible for all errors.

I am also happy to acknowledge that I benefited from material published by colleagues on diverse topics that are partially covered in this lecture. While some of the material used for this lecture was originally presented in traditional-style publications (such as textbooks), some other material has its roots in non-standard publication outlets (such as online documentations, electronic course notes, or user manuals).

Salzburg, July 2020

Martin Held
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Recommended Textbooks I

G. Farin.  
*Curves and Surfaces for CAGD: A Practical Guide.*  

R.H. Bartels, J.C. Beatty, B.A. Barsky.  
*An Introduction to Splines for Use in Computer Graphics and Geometric Modeling.*  

H. Prautzsch, W. Boehm, M. Paluszny.  
*Bézier and B-spline Techniques.*  

J. Gallier.  
*Curves and Surfaces in Geometric Modeling.*  
http://www.cis.upenn.edu/~jean/gbooks/geom1.html

R. Goldman.  
Recommended Textbooks II

N.M. Patrikalakis, T. Maekawa, W. Cho.
*Shape Interrogation for Computer Aided Design and Manufacturing.*
http://web.mit.edu/hyperbook/Patrikalakis-Maekawa-Cho/

M. Botsch, L. Kobbelt, M. Pauly, P. Alliez, B. Levy.
*Polygon Mesh Processing.*
http://www.pmp-book.org/

G.E. Farin, D. Hansford.
*Practical Linear Algebra: A Geometry Toolbox.*

M.E. Mortenson.

A. Dickenstein, I.Z. Emiris (eds.).
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Motivation: Evaluation of a Polynomial

- Assume that we have an intuitive understanding of polynomials and consider a polynomial in $x$ of degree $n$ with coefficients $a_0, a_1, \ldots, a_n \in \mathbb{R}$, with $a_n \neq 0$:

$$p(x) := \sum_{i=0}^{n} a_i x^i = a_0 + a_1 x + a_2 x^2 + \ldots + a_{n-1} x^{n-1} + a_n x^n.$$  

- A straightforward polynomial evaluation of $p$ for a given parameter $x_0$ — i.e., the computation of $p(x_0)$ — results in $k$ multiplications for a monomial of degree $k$, plus a total of $n$ additions.

- Hence, we would get

$$0 + 1 + 2 + \ldots + n = \frac{n(n+1)}{2} = O(n^2)$$

multiplications (and $n$ additions).

- Can we do better?

- Yes, we can: Horner’s Algorithm consumes only $n$ multiplications and $n$ additions to evaluate a polynomial of degree $n$!
Motivation: Smoothness of a Curve

What is a characteristic difference between the three curves shown below?

Answer: The green curve has tangential discontinuities at the vertices, the blue curve consists of straight-line segments and circular arcs and is tangent-continuous, while the red curve is a cubic B-spline and is curvature-continuous.

By the way, when precisely is a geometric object a “curve”? 
Motivation: Tangent to a Curve

- What is a parametrization of the tangent line at a point \( \gamma(t_0) \) of a curve \( \gamma \)?

- Answer: If \( \gamma \) is differentiable then a parametrization of the tangent line \( \ell \) that passes through \( \gamma(t_0) \) is given by

\[
\ell(\lambda) = \gamma(t_0) + \lambda \gamma'(t_0) \quad \text{with} \quad \lambda \in \mathbb{R}.
\]

- How can we obtain \( \gamma'(t) \) for \( \gamma: \mathbb{R} \to \mathbb{R}^d \)?
Motivation: Bézier Curve

- How can we model a “smooth” polynomial curve in $\mathbb{R}^2$ by specifying so-called “control points”. (E.g., the points $p_0, p_1, \ldots, p_{10}$ in the figure.)

One (widely used) option is to generate a Bézier curve. (The figure shows a Bézier curve of degree 10 with 11 control points.)

- What is the degree of a Bézier curve? Which geometric and mathematical properties do Bézier curves exhibit?
Motivation: B-Spline Curve

- How can we model a (piecewise) polynomial curve in $\mathbb{R}^2$ by specifying so-called “control points” such that a modification of one control point affects only a “small” portion of the curve?

- Answer: Use B-spline curves.

- Which geometric and mathematical properties do B-spline curves exhibit?
Motivation: NURBS

- Is it possible to parameterize a circular arc by means of a polynomial term? Or by a rational term?
- Yes, this is possible by means of a rational term:
  \[
  \left( \frac{1 - t^2}{1 + t^2}, \frac{2t}{1 + t^2} \right)
  \]
  for \( t \in \mathbb{R} \).
- More generally, NURBS can be used to model all types of conics by means of rational parametrizations.
Motivation: Approximation of a Continuous Function

- How can we approximate a continuous function by a polynomial?
- Answer: We can use a Bernstein approximation.
- Sample Bernstein approximations of a continuous function:

\[ f : [0, 1] \rightarrow \mathbb{R} \quad f(x) := \sin(\pi x) + \frac{1}{5}\sin(6\pi x + \pi x^2) \]

- One can prove that the Bernstein approximation \( B_{n,f} \) converges uniformly to \( f \) on the interval \([0, 1]\) as \( n \) increases, for every continuous function \( f \).
Notation: Numbers and Sets

- Numbers:
  - The set \{1, 2, 3, \ldots\} of natural numbers is denoted by \(\mathbb{N}\), with \(\mathbb{N}_0 := \mathbb{N} \cup \{0\}\).
  - The set \{2, 3, 5, 7, 11, 13, \ldots\} \subset \mathbb{N} of prime numbers is denoted by \(\mathbb{P}\).
  - The (positive and negative) integers are denoted by \(\mathbb{Z}\).
  - \(\mathbb{Z}_n := \{0, 1, 2, \ldots, n – 1\}\) and \(\mathbb{Z}_n^+ := \{1, 2, \ldots, n – 1\}\) for \(n \in \mathbb{N}\).
  - The reals are denoted by \(\mathbb{R}\); the non-negative reals are denoted by \(\mathbb{R}_0^+\), and the positive reals by \(\mathbb{R}^+\).

- Open or closed intervals \(I \subset \mathbb{R}\) are denoted using square brackets: e.g., \(I_1 = [a_1, b_1]\) or \(I_2 = [a_2, b_2]\), with \(a_1, a_2, b_1, b_2 \in \mathbb{R}\), where the right-hand “[” indicates that the value \(b_2\) is not included in \(I_2\).

- The set of all elements \(a \in A\) with property \(P(a)\), for some set \(A\) and some predicate \(P\), is denoted by

\[
\{x \in A : P(x)\} \quad \text{or} \quad \{x : x \in A \land P(x)\}
\]

or

\[
\{x \in A \mid P(x)\} \quad \text{or} \quad \{x \mid x \in A \land P(x)\}.
\]

- Quantifiers: The universal quantifier is denoted by \(\forall\), and \(\exists\) denotes the existential quantifier.

- Bold capital letters, such as \(\mathbf{M}\), are used for matrices.

- The set of all (real) \(m \times n\) matrices is denoted by \(\mathbf{M}_{m \times n}\).
Notation: Vectors

- Points are denoted by letters written in italics: \( p, q \) or, occasionally, \( P, Q \). We do not distinguish between a point and its position vector.
- The coordinates of a vector are denoted by using indices (or numbers): e.g., \( \mathbf{v} = (v_x, v_y) \) for \( \mathbf{v} \in \mathbb{R}^2 \), or \( \mathbf{v} = (v_1, v_2, \ldots, v_n) \) for \( \mathbf{v} \in \mathbb{R}^n \).
- In order to state \( \mathbf{v} \in \mathbb{R}^n \) in vector form we will mix column and row vectors freely unless a specific form is required, such as for matrix multiplication.
- The vector dot product of two vectors \( \mathbf{v}, \mathbf{w} \in \mathbb{R}^n \) is denoted by \( \langle \mathbf{v}, \mathbf{w} \rangle \). That is, \( \langle \mathbf{v}, \mathbf{w} \rangle = \sum_{i=1}^{n} v_i \cdot w_i \) for \( \mathbf{v}, \mathbf{w} \in \mathbb{R}^n \).
- The vector cross-product (in \( \mathbb{R}^3 \)) is denoted by a cross: \( \mathbf{v} \times \mathbf{w} \).
- The length of a vector \( \mathbf{v} \) is denoted by \( \| \mathbf{v} \| \).
- The straight-line segment between the points \( p \) and \( q \) is denoted by \( \overline{pq} \).
- The supporting line of the points \( p \) and \( q \) is denoted by \( \ell(p, q) \).
Consider $k$ real numbers $a_1, a_2, \ldots, a_k \in \mathbb{R}$, together with some $m, n \in \mathbb{N}$ such that $1 \leq m, n \leq k$.

\[
\sum_{i=m}^{n} a_i := \begin{cases} 
0 & \text{if } n < m \\
 a_m & \text{if } n = m \\
(\sum_{i=m}^{n-1} a_i) + a_n & \text{if } n > m 
\end{cases}
\]

\[
\prod_{i=m}^{n} a_i := \begin{cases} 
1 & \text{if } n < m \\
 a_m & \text{if } n = m \\
(\prod_{i=m}^{n-1} a_i) \cdot a_n & \text{if } n > m 
\end{cases}
\]
Mathematics for Geometric Modeling
Factorial and Binomial Coefficient
Polynomials
Elementary Differential Calculus
Elementary Differential Geometry of Curves
Elementary Differential Geometry of Surfaces
Factorial and Binomial Coefficient

Definition 1 (*Factorial, Dt.: Fakultät, Faktorielle*)

For \( n \in \mathbb{N}_0 \),
\[
  n! := \begin{cases} 
    1 & \text{if } n \leq 1, \\
    n \cdot (n - 1)! & \text{if } n > 1.
  \end{cases}
\]

▶ Note that \( 0! = 1 \) by definition!

Definition 2 (*Binomial coefficient, Dt.: Binomialkoeffizient*)

Let \( n \in \mathbb{N}_0 \) and \( k \in \mathbb{Z} \). Then the *binomial coefficient* \( \binom{n}{k} \) of \( n \) and \( k \) is defined as follows:
\[
  \binom{n}{k} := \begin{cases} 
    0 & \text{if } k < 0, \\
    \frac{n!}{k! \cdot (n-k)!} & \text{if } 0 \leq k \leq n, \\
    0 & \text{if } k > n.
  \end{cases}
\]

▶ The binomial coefficient \( \binom{n}{k} \) is pronounced as “\( n \) choose \( k \)”; Dt.: “\( n \) über \( k \)”. 

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Factorial and Binomial Coefficient

Lemma 3

Let $n \in \mathbb{N}_0$ and $k \in \mathbb{Z}$.

\[
\binom{n}{0} = \binom{n}{n} = 1 \quad \binom{n}{1} = \binom{n}{n-1} = n \quad \binom{n}{k} = \binom{n}{n-k}
\]

Theorem 4 (Khayyam, Yang Hui, Tartaglia, Pascal)

For $n \in \mathbb{N}$ and $k \in \mathbb{Z}$,

\[
\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.
\]
Factorial and Binomial Coefficient

Theorem 5 (Binomial Theorem, Dt.: Binomischer Lehrsatz)

For all $n \in \mathbb{N}_0$ and $a, b \in \mathbb{R}$,

$$(a + b)^n = \binom{n}{0} a^n + \binom{n}{1} a^{n-1} b + \cdots + \binom{n}{n} b^n$$

or, equivalently,

$$(a + b)^n = \sum_{i=0}^{n} \binom{n}{i} a^{n-i} b^i.$$

In particular, for all $a, b \in \mathbb{R}$,

$$(a + b)^2 = a^2 + 2ab + b^2 \quad (a + b)^3 = a^3 + 3a^2 b + 3ab^2 + b^3.$$
Polynomials

Definition 6 (Monomial, Dt.: Monom)
A (real) monomial in \( m \) variables \( x_1, x_2, \ldots, x_m \) is a product of a coefficient \( c \in \mathbb{R} \) and powers of the variables \( x_i \) with exponents \( k_i \in \mathbb{N}_0 \):

\[
c \prod_{i=1}^{m} x_i^{k_i} = c \cdot x_1^{k_1} \cdot x_2^{k_2} \cdot \ldots \cdot x_m^{k_m}.
\]

The degree of the monomial is given by \( \sum_{i=1}^{m} k_i \).

Definition 7 (Polynomial, Dt.: Polynom)
A (real) polynomial in \( m \) variables \( x_1, x_2, \ldots, x_m \) is a finite sum of monomials in \( x_1, x_2, \ldots, x_m \).

A polynomial is univariate if \( m = 1 \), bivariate if \( m = 2 \), and multivariate otherwise.

Definition 8 (Degree, Dt.: Grad)
The degree of a polynomial is the maximum degree of its monomials.
Hence, a univariate polynomial over $\mathbb{R}$ with variable $x$ is a term of the form

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0,$$

with coefficients $a_0, \ldots, a_n \in \mathbb{R}$ and $a_n \neq 0$.

It is a convention to drop all monomials whose coefficients are zero.

Univariate polynomials are usually written according to a decreasing order of exponents of their monomials.

In that case, the first term is the leading term which indicates the degree of the polynomial; its coefficient is the leading coefficient.

Univariate polynomials of degree

0. are called constant polynomials,
1. are called linear polynomials,
2. are called quadratic polynomials,
3. are called cubic polynomials,
4. are called quartic polynomials,
5. are called quintic polynomials.
Polynomial Arithmetic

- We define the addition of (univariate) polynomials based on the pairwise addition of corresponding coefficients:

\[
\left( \sum_{i=0}^{n} a_i x^i \right) + \left( \sum_{i=0}^{n} b_i x^i \right) := \sum_{i=0}^{n} (a_i + b_i) x^i
\]

- The multiplication of polynomials is based on the multiplication within \( \mathbb{R} \), distributivity, and the rules

\[
ax = xa \quad \text{and} \quad x^m x^k = x^{m+k}
\]

for all \( a \in \mathbb{R} \) and \( m, k \in \mathbb{N} \):

\[
\left( \sum_{i=0}^{n} a_i x^i \right) \cdot \left( \sum_{j=0}^{m} b_j x^j \right) := \sum_{i=0}^{n} \sum_{j=0}^{m} (a_i b_j) x^{i+j}
\]

- Elementary properties of polynomials: One can prove easily that the addition, multiplication and composition of two polynomials as well as their derivative and antiderivative (indefinite integral) again yield a polynomial.

- Same for multivariate polynomials.
Polynomial Arithmetic

- Instead of $\mathbb{R}$ any commutative ring $(R, +, \cdot)$ and symbols $x, y, \ldots$ that are not contained in $R$ would do. E.g.,

$$a_{2,3}x^2 y^3 + a_{1,1}xy + a_{0,1}y + a_{0,0} \quad \text{with } a_{2,3}, a_{1,1}, a_{0,1}, a_{0,0} \in R.$$

Lemma 9

The set of all polynomials with coefficients in the commutative ring $(R, +, \cdot)$ and a symbol (variable) $x \not\in R$ forms a commutative ring, the ring of polynomials over $R$, commonly denoted by $R[x]$.

- Multivariate polynomials can also be seen as univariate polynomials with coefficients out of a ring of polynomials. E.g.,

$$a_{2,3}x^2 y^3 + a_{1,1}xy + a_{0,1}y + a_{0,0} = (a_{2,3}x^2)y^3 + (a_{1,1}x + a_{0,1})y + a_{0,0}$$

is an element of $R[x, y] := (R[x])[y]$.

Definition 10

Two polynomials are equal if and only if the sequences of their coefficients (arranged in some specific order) are equal.
The univariate polynomials of $\mathbb{R}[x]$ form an infinite vector space over $\mathbb{R}$. The so-called power basis of this vector space is given by the monomials $1, x, x^2, x^3, \ldots$.

- The monomials $1, x, x^2, x^3, \ldots, x^n$ form a basis of the vector space of polynomials of degree up to $n$ over $\mathbb{R}$, for all $n \in \mathbb{N}_0$.

- Recall: The fact that the monomials $1, x, x^2, x^3, \ldots, x^n$ form a basis of the polynomials of degree up to $n$ over $\mathbb{R}$ means that

1. every polynomial $p \in \mathbb{R}[x]$ of degree at most $n$ can be expressed as a linear combination of those monomials: there exist $a_0, a_1, \ldots, a_n \in \mathbb{R}$ such that

$$p = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0,$$

2. none of those monomials can be expressed as a linear combination of the other monomials, i.e., the monomials are linearly independent.

- The power basis is not the only meaningful basis of the polynomials $\mathbb{R}[x]$. See, e.g., the Bernstein polynomials that are used to form Bézier curves.
**Polynomials: Roots**

**Definition 12 (Polynomial equation)**

A *polynomial equation* (aka *algebraic equation*) is an equation in which a polynomial is set equal to another polynomial.

**Definition 13 (Root, Dt.: Wurzel)**

The polynomial \( p \in \mathbb{R}[x] \) has a *root* (aka *zero*) \( r \in \mathbb{R} \) if \((x - r)\) divides \( p \).

► Hence, if \( r \) is a root of \( p \) then \( p = (x - r) \cdot p_1 \) for some \( p_1 \in \mathbb{R}[x] \).

**Definition 14 (Multiplicity, Dt.: Vielfachheit)**

A root \( r \) of a polynomial \( p \in \mathbb{R}[x] \) is of *multiplicity* \( k \) if \( k \in \mathbb{N} \) is the maximum integer such that \((x - r)^k\) divides \( p \).

**Theorem 15 (Fundamental Theorem of Algebra)**

The number of complex roots of a polynomial with real coefficients may not exceed its degree. It equals the degree if all roots are counted with their multiplicities.
Recall the quadratic formula taught in secondary school for solving second-degree polynomial equations: For $a \in \mathbb{R} \setminus \{0\}$ and $b, c \in \mathbb{R}$,

\[
    x_{1,2} := \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\]

yields the two (possibly complex) roots $x_1$ and $x_2$ of $ax^2 + bx + c$.

Similar (albeit more complex) formulas exist for cubic and quartic polynomials.

**Theorem 16 (Abel-Ruffini (1824))**

No algebraic solution for the roots of an arbitrary polynomial of degree five or higher exists.

An algebraic solution is a closed-form expressions in terms of the coefficients of the polynomial that relies only on addition, subtraction, multiplication, division, raising to integer powers, and computing $k$-th roots (square roots, cube roots, and other integer roots).

A closed-form expression is an expression that can be evaluated in a finite number of operations.
Polynomials: Roots

Lemma 17

For \(a, b, c \in \mathbb{R}\), the roots \(r_1, r_2\) of the quadratic polynomial \(ax^2 + bx + c\) satisfy
\[
\begin{align*}
    r_1 + r_2 &= -\frac{b}{a} \\
    r_1 \cdot r_2 &= \frac{c}{a}.
\end{align*}
\]

Lemma 18

For \(a, b, c, d \in \mathbb{R}\), the roots \(r_1, r_2, r_3\) of the cubic polynomial \(ax^3 + bx^2 + cx + d\) satisfy
\[
\begin{align*}
    r_1 + r_2 + r_3 &= -\frac{b}{a} \\
    r_1 \cdot r_2 + r_1 \cdot r_3 + r_2 \cdot r_3 &= \frac{c}{a} \\
    r_1 \cdot r_2 \cdot r_3 &= -\frac{d}{a}.
\end{align*}
\]

These two lemmas are special cases of a general theorem by François Viète (Franciscus Vieta, 1540–1603).
Polynomials: Function

Definition 19 (*Polynomial function*; Dt.: *Polynomfunktion*)

A (univariate real) function \( f : I \to \mathbb{R} \), for an interval \( I \subseteq \mathbb{R} \), is a polynomial function over \( I \) if there exist \( n \in \mathbb{N}_0 \) and \( a_0, a_1, \ldots, a_n \in \mathbb{R} \) such that

\[
f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \quad \text{for all } x \in I.
\]

- As usual, two (polynomial) functions over an interval \( I \subseteq \mathbb{R} \) are identical if their values are identical for all arguments in \( I \).
- Note: Two different polynomials may result in the same polynomial function! (E.g., over finite fields.)
- While some fields of mathematics (e.g., abstract algebra) make a clear distinction between polynomials and polynomial functions, we will freely mix these two terms. Also, unless noted explicitly, we will only deal with polynomials over \( \mathbb{R} \).
- Note: Polynomial functions may come in disguise: \( f(x) := \cos(2 \arccos(x)) \) is a polynomial function over \([-1, 1]\), since we have \( f(x) = 2x^2 - 1 \) for all \( x \in [-1, 1] \).
Consider a polynomial $p \in \mathbb{R}[x]$ of degree $n$ with coefficients $a_0, a_1, \ldots, a_n \in \mathbb{R}$, with $a_n \neq 0$:

$$p(x) := \sum_{i=0}^{n} a_i x^i = a_0 + a_1 x + a_2 x^2 + \ldots + a_{n-1} x^{n-1} + a_n x^n.$$ 

A straightforward polynomial evaluation of $p$ for a given parameter $x_0$ results in $k$ multiplications for a monomial of degree $k$, plus a total of $n$ additions.

Hence, we would get

$$0 + 1 + 2 + \ldots + n = \frac{n(n+1)}{2}$$

multiplications (and $n$ additions).

Can we do better?

Obviously, we can reduce the number of multiplications to $O(n \log n)$ by resorting to exponentiation by squaring:

$$x^n = \begin{cases} 
  x \left( x^2 \right)^{\frac{n-1}{2}} & \text{if } n \text{ is odd}, \\
  \left( x^2 \right)^{\frac{n}{2}} & \text{if } n \text{ is even}.
\end{cases}$$

Can we do even better?
Horner’s Algorithm for Evaluation of a Polynomial

- **Horner’s Algorithm**: The idea is to rewrite the polynomial such that

\[
p(x) = a_0 + x\left(a_1 + x\left(a_2 + \ldots + x(a_{n-2} + x(a_{n-1} + x a_n))\ldots \right)\right)
\]

and compute the result \( h_0 := p(x_0) \) as follows:

\[
\begin{align*}
    h_n & := a_n \\
    h_i & := x_0 \cdot h_{i+1} + a_i \quad \text{for } i = 0, 1, 2, \ldots, n - 1
\end{align*}
\]

```c
/** Evaluates a polynomial of degree n at point x */
#define evaluate(p, n, x) 
{
    double h = p[n];
    for (int i = n - 1; i >= 0; --i)
        h = x * h + p[i];
    return h;
}
```
Horner’s Algorithm for Evaluation of a Polynomial

Lemma 20
Horner’s Algorithm consumes $n$ multiplications and $n$ additions to evaluate a polynomial of degree $n$.

Caveat
Subtractive cancellation could occur at any time, and there is no easy way to determine a priori whether and for which data it will indeed occur.

- Subtractive cancellation: Subtracting two nearly equal numbers (on a conventional IEEE-754 floating-point arithmetic) may yield a result with few or no meaningful digits. Aka: catastrophic cancellation.
Forward Differencing

- If a polynomial has to be evaluated for \(k + 1\) evenly spaced arguments \(x_0, x_1, \ldots, x_k\), with \(x_{i+1} = x_i + \delta\) for \(0 \leq i < k\), then *forward differencing* is faster than Horner’s Algorithm.

- Consider a polynomial of degree one:

\[
p(x) = a_0 + a_1 x
\]

- The difference in the function values \(p(x_i)\) and \(p(x_{i+1})\) of two neighboring points \(x_i\) and \(x_{i+1}\) is

\[
\Delta := p(x_{i+1}) - p(x_i) = p(x_i + \delta) - p(x_i) = a_0 + a_1(x_i + \delta) - (a_0 + a_1 x_i) = a_1 \delta.
\]

- Hence, to evaluate the polynomial for several arguments, we may start with the evaluation of \(p(x_0)\) and recursively compute

\[
p(x_{i+1}) = p(x_i + \delta) = p(x_i) + \Delta, \quad \text{with } \Delta := a_1 \delta.
\]
For a quadratic polynomial $p(x) = a_0 + a_1 x + a_2 x^2$ the difference of the function values of neighboring arguments is

$$\Delta_1(x_i) := p(x_{i+1}) - p(x_i) = p(x_i + \delta) - p(x_i)$$

$$= a_0 + a_1 (x_i + \delta) + a_2 (x_i + \delta)^2 - (a_0 + a_1 x_i + a_2 x_i^2)$$

$$= a_1 \delta + a_2 \delta^2 + 2 a_2 x_i \delta$$

As $\Delta_1(x)$ is itself a linear polynomial, it can also be evaluated using forward differencing:

$$\Delta_2(x) := \Delta_1(x + \delta) - \Delta_1(x) = 2 a_2 \delta^2$$

This approach can be extended to polynomials of any degree.

One can conclude that for a polynomial of degree $n$ each successive evaluation requires $n$ additions.
Differentiation of Functions of One Variable

Definition 21 (*Derivative, Dt.: Ableitung*)

Let $S \subseteq \mathbb{R}$ be an open set. A (scalar-valued) function $f : S \to \mathbb{R}$ is *differentiable* at an interior point $x_0 \in S$ if

$$
\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}
$$

exists, in which case the limit is called the *derivative* of $f$ at $x_0$, denoted by $f'(x_0)$.

Definition 22

Let $S \subseteq \mathbb{R}$ be an open set. A (scalar-valued) function $f : S \to \mathbb{R}$ is *differentiable on* $S$ if it is differentiable at every point of $S$. If $f$ is differentiable on $S$ and $f'$ is continuous on $S$ then $f$ is *continuously differentiable on* $S$. In this case $f$ is said to be of differentiability class $C^1$.

* By taking one-sided limits one can also consider one-sided derivatives on the boundary of closed sets $S$.

* By applying differentiation to $f'$, a second derivative $f''$ of $f$ can be defined. Inductively, we obtain $f^{(n)}$ by differentiating $f^{(n-1)}$. 
Differentiation of Functions of One Variable

Definition 23 ($C^k$, Dt.: $k$-mal stetig differenzierbar)

Let $S \subseteq \mathbb{R}$ be an open set. A function $f : S \rightarrow \mathbb{R}$ that has $k$ successive derivatives is called $k$ times differentiable. If, in addition, the $k$-th derivative is continuous, then the function is said to be of differentiability class $C^k$.

- If the $k$-th derivative of $f$ exists then the continuity of $f^{(0)}, f^{(1)}, \ldots, f^{(k-1)}$ is implied.

Definition 24 (Smooth, Dt.: glatt)

Let $S \subseteq \mathbb{R}$ be an open set. A function $f : S \rightarrow \mathbb{R}$ is called smooth if it has infinitely many derivatives, denoted by the class $C^\infty$.

- We have $C^\infty \subset C^i \subset C^j$, for all $i, j \in \mathbb{N}_0$ if $i > j$.

- Notation:
  - $f^{(0)}(x) := f(x)$ for convenience purposes.
  - $f'(x) = f^{(1)}(x) = \frac{d}{dx} f(x) = \frac{df}{dx}(x)$.
  - $f''(x) = f^{(2)}(x) = \frac{d^2}{dx^2} f(x) = \frac{d^2 f}{dx^2}(x)$.
  - $f'''(x) = f^{(3)}(x) = \frac{d^3}{dx^3} f(x) = \frac{d^3 f}{dx^3}(x)$.
  - $f^{(n)}(x) = \frac{d^n}{dx^n} f(x) = \frac{d^n f}{dx^n}(x)$.
Definition 25

For $n \in \mathbb{N}$ consider $n$ functions $f_i : S \to \mathbb{R}$ (with $1 \leq i \leq n$) and define $f : S \to \mathbb{R}^n$ as

$$f(x) := \begin{pmatrix} f_1(x) \\ f_2(x) \\ \vdots \\ f_n(x) \end{pmatrix}.$$

Then the (vector-valued) function $f$ is differentiable at an interior point $x_0 \in S$ if and only if $f_i$ is differentiable at $x_0$, for all $i \in \{1, 2, \ldots, n\}$. The derivative of $f$ at $x_0$ is given by

$$f'(x_0) := \begin{pmatrix} f'_1(x_0) \\ f'_2(x_0) \\ \vdots \\ f'_n(x_0) \end{pmatrix}.$$

All other definitions related to differentiability carry over from scalar-valued functions to vector-valued functions of one variable in a natural way.
Differentiation of Functions of Several Variables

Definition 26 *(Partial derivative, Dt.: partielle Ableitung)*

Let \( S \subseteq \mathbb{R}^m \) be an open set. The *partial derivative* of a (vector-valued) function \( f : S \rightarrow \mathbb{R}^n \) at point \( (a_1, a_2, \ldots, a_m) \in S \) with respect to the \( i \)-th coordinate \( x_i \) is defined as

\[
\frac{\partial f}{\partial x_i}(a_1, a_2, \ldots, a_m) := \lim_{h \to 0} \frac{f(a_1, a_2, \ldots, a_i + h, \ldots, a_m) - f(a_1, a_2, \ldots, a_i, \ldots, a_m)}{h},
\]

if this limit exists.

- Hence, for a partial derivative with respect to \( x_i \) we simply differentiate \( f \) with respect to \( x_i \) according to the rules for ordinary differentiation, while regarding all other variables as constants.
- That is, for the purpose of the partial derivative with respect to \( x_i \) we regard \( f \) as univariate function in \( x_i \) and apply standard differentiation rules.
- Some authors prefer to write \( f_x \) instead of \( \frac{\partial f}{\partial x} \).
- We will mix notations as we find it convenient.

**Note**

A function of \( m \) variables may have all first-order partial derivatives at a point \( (a_1, \ldots, a_m) \) but still need not be continuous at \( (a_1, \ldots, a_m) \).
Differentiation of Functions of Several Variables

- Higher-order partial derivatives of $f$ are obtained by repeated computation of a partial derivative of a (higher-order) partial derivative of $f$.
- Does it matter in which sequence we compute higher-order partial derivatives?

Theorem 27

Suppose that $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, $\frac{\partial^2 f}{\partial y \partial x}$ exist on a neighborhood of $(x_0, y_0)$, and that $\frac{\partial^2 f}{\partial y \partial x}$ is continuous at $(x_0, y_0)$. Then $\frac{\partial^2 f}{\partial x \partial y}(x_0, y_0)$ exists, and we have

$$\frac{\partial^2 f}{\partial x \partial y}(x_0, y_0) = \frac{\partial^2 f}{\partial y \partial x}(x_0, y_0).$$

- Note the difference between the Leibniz notation and the subscript notation for higher-order mixed partial derivatives!

$$\frac{\partial^2 f}{\partial x \partial y}(x, y) := \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial y} \right)(x, y) \quad \text{but} \quad f_{xy}(x, y) := (f_x)_y(x, y)$$

- Also, note that Theorem 27 does not imply that it could not be simpler to compute, say, $\frac{\partial^2 f}{\partial y \partial x}$ rather than $\frac{\partial^2 f}{\partial x \partial y}$:

$$f(x, y) := xe^{2y} + \sqrt{e^y \sin(y \tan(\log y))} + \sqrt{1 + y^2 \cos^2 y}$$
Definition 28 (Differentiable, Dt.: total differenzierbar)

Let $S \subseteq \mathbb{R}^m$ be an open set. A function $f : S \to \mathbb{R}^n$ of $m$ variables is differentiable at a point $a := (a_1, \ldots, a_m) \in S$ if there exists an $n \times m$ matrix $J$ such that

$$\lim_{x \to a} \frac{f(x) - f(a) - J(x - a)}{\|x - a\|} = 0.$$ 

Theorem 29

Let $S \subseteq \mathbb{R}^m$ be an open set. If a function $f : S \to \mathbb{R}^n$ of $m$ variables is differentiable at a point $a \in S$ then the coefficients $a_{ij}$ of the matrix $J$ of Def. 28 are given by

$$a_{ij} = \frac{\partial f_i}{\partial x_j}(a) \quad \text{for } i \in \{1, 2, \ldots, n\} \text{ and } j \in \{1, 2, \ldots, m\}.$$ 

The matrix $J$ is called Jacobi matrix of $f$ at $a$. 

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Differentiation of Functions of Several Variables

**Theorem 30**

If a function \( f: S \rightarrow \mathbb{R}^n \) of \( m \) variables is differentiable at a point \( a \in S \) then it is continuous at \( a \).

**Theorem 31**

If \( \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_m} \) exist for a function \( f: S \rightarrow \mathbb{R}^n \) of \( m \) variables on a neighborhood of a point \( a \in S \) and are continuous at \( a \) then \( f \) is differentiable at \( a \).

**Definition 32** *(Continuously differentiable, Dt.: stetig differenzierbar)*

We say that a function \( f: S \rightarrow \mathbb{R}^n \) of \( m \) variables is *continuously differentiable* on an open subset \( S \) of \( \mathbb{R}^m \) if \( \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_m} \) exist and are continuous on \( S \).
Intuitively, a curve in $\mathbb{R}^2$ is generated by a continuous motion of a pencil on a sheet of paper.

A formal mathematical definition is not entirely straightforward, and the term “curve” is associated with two closely related notions: kinematic and geometric.

In the kinematic setting, a (parameterized) curve is a function of one real variable.

In the geometric setting, a curve, also called an arc, is a 1-dimensional subset of space that is “similar” to a line (albeit it need not be straight).

Both notions are related:

- The image of a parameterized curve describes an arc.
- Conversely, an arc admits a parametrization.

Since the kinematic setting is easier to introduce, we resort to a kinematic definition of “curve”.

Note that fairly counter-intuitive curves exist: e.g., space-filling curves like the Sierpinski curve.
Caveat: Sierpinski Curves

- Sierpinski curves are a sequence of recursively defined continuous and closed curves in \( \mathbb{R}^2 \).
- Sierpinski curve of orders 1–3:

Their limit curve, the Sierpinski curve, is a space-filling curve: It fills the unit square completely! It is a continuous and surjective (but not injective!) mapping of \([0, 1]\) onto \([0, 1] \times [0, 1]\).

- The Euclidean length \( L_n \) of the order-\( n \) Sierpinski curve is

\[
L_n = \frac{2}{3} (1 + \sqrt{2}) 2^n - \frac{1}{3} (2 - \sqrt{2}) \frac{1}{2^n}.
\]

Hence, its length grows exponentially and unboundedly as \( n \) grows.
Curves in $\mathbb{R}^n$

**Definition 33 (Curve, Dt.: Kurve)**

Let $I \subseteq \mathbb{R}$ be an interval of the real line. A continuous (vector-valued) mapping $\gamma: I \rightarrow \mathbb{R}^n$ is called a *parametrization* of $\gamma(I)$ or a *parametric curve*.

- Well-known examples of parameterized curves include a straight-line segment, a circular arc, and a helix.
- E.g., $\gamma: [0, 1] \rightarrow \mathbb{R}^3$ with

$$
\gamma(t) := \begin{pmatrix}
    p_x + t \cdot (q_x - p_x) \\
    p_y + t \cdot (q_y - p_y) \\
    p_z + t \cdot (q_z - p_z)
\end{pmatrix}
$$

maps $[0, 1]$ to a straight-line segment from point $p$ to $q$.

- The interval $I$ is called the *domain* of $\gamma$, and $\gamma(I)$ is called *image* (Dt.: Bild, Spur).

**Definition 34 (Plane curve, Dt.: ebene Kurve)**

For $\gamma: I \rightarrow \mathbb{R}^n$, the curve $\gamma(I)$ is *plane* if $\gamma(I) \subseteq \mathbb{R}^2$ or if $\gamma(I)$ lies within an affine/projective plane. A non-plane curve is called a *skew curve* (Dt.: Raumkurve).

- An *algebraic plane curve* is the zero set of a polynomial in two variables.
Curves in $\mathbb{R}^n$

Definition 35 (*Start and end point*)

If $I$ is a closed interval $[a, b]$, for some $a, b \in \mathbb{R}$, then we call $\gamma(a)$ the **start point** and $\gamma(b)$ the **end point** of the curve $\gamma: I \to \mathbb{R}^n$.

Definition 36 (*Closed, Dt.: geschlossen*)

A parametrization $\gamma: I \to \mathbb{R}^n$ is said to be **closed** (or a **loop**) if $I$ is a closed interval $[a, b]$, for some $a, b \in \mathbb{R}$, and $\gamma(a) = \gamma(b)$.

Definition 37 (*Simple, Dt.: einfach*)

A parametrization $\gamma: I \to \mathbb{R}^n$ is said to be **simple** if $\gamma(t_1) = \gamma(t_2)$ for $t_1 \neq t_2 \in I$ implies $\{t_1, t_2\} = \{a, b\}$ and $I = [a, b]$, for some $a, b \in \mathbb{R}$.

▶ Hence, if $\gamma: I \to \mathbb{R}^n$ is simple then it is injective on $\text{int}(I)$.
Curves in $\mathbb{R}^n$

- Many properties of curves can also be stated independently of a specific parametrization. E.g., we can regard a curve $C$ to be simple if there exists one parametrization of $C$ that is simple.

- In daily math, the standard meaning of a “curve” is the image of the equivalence class of all paths under a certain equivalence relation. (Roughly, two paths are equivalent if they are identical up to re-parametrization.)

- Hence, the distinction between a curve and (one of) its parametrizations is often blurred.

- For the sake of simplicity, we will not distinguish between a curve $C$ and one of its parametrizations $\gamma$ if the meaning is clear.

- Similarly, we will frequently call $\gamma$ a curve.

- For instance, we will frequently speak about a closed curve rather than about a closed parametrization of a curve.
Jordan Curve in $\mathbb{R}^2$

**Definition 38 (Jordan curve)**

A set $C \subset \mathbb{R}^2$ (which is not a single point) is called a *Jordan curve* if there exists a simple and closed parametrization $\gamma : I \to \mathbb{R}^2$ that parameterizes $C$.

**Theorem 39 (Jordan 1887)**

Every Jordan curve $C$ partitions $\mathbb{R}^2 \setminus C$ into two disjoint open regions, a (bounded) “interior” region and an (unbounded) “exterior” region, with $C$ as the (topological) boundary of both of them.

▶ Although this theorem — the so-called Jordan Curve Theorem (Dt.: Jordanscher Kurvensatz) — seems obvious, a proof is not entirely trivial.

**Theorem 40 (Schönflies 1906)**

For every Jordan curve $C$ there exists a homeomorphism from the plane to itself that maps $C$ to the unit sphere $S^1$.

▶ Roughly, a homeomorphism is a bijective continuous stretching and bending of one space into another space such that the inverse function also is continuous.
Differentiable Curves

Definition 41 \((C^r\text{-parametrization})\)

If \(\gamma: I \to \mathbb{R}^n\) is \(r\) times continuously differentiable then \(\gamma\) is called a parametric curve of class \(C^r\), or a \(C^r\text{-parametrization}\) of \(\gamma(I)\), or simply a \(C^r\text{-curve}\).

If \(I = [a, b]\), then \(\gamma\) is called a **closed \(C^r\text{-parametrization}** if \(\gamma^{(k)}(a) = \gamma^{(k)}(b)\) for all \(0 \leq k \leq r\).

▶ One-sided differentiability is meant at the endpoints of \(I\) if \(I\) is a closed interval.

Definition 42 \((\text{Smooth curve}, \text{Dt.: glatte Kurve})\)

If \(\gamma: I \to \mathbb{R}^n\) has derivatives of all orders then \(\gamma\) is (the parametrization of) a **smooth curve** (or of class \(C^\infty\)).

Definition 43 \((\text{Piecewise smooth curve}, \text{Dt.: stückweise glatte Kurve})\)

If \(I\) is the union of a finite number of sub-intervals over each of which \(\gamma: I \to \mathbb{R}^n\) is smooth then \(\gamma\) is **piecewise smooth**.

▶ Note: Smoothness depends on the parametrization!

▶ There do exist curves which are continuous everywhere but differentiable nowhere [Weierstrass 1872, Koch 1904].
Differentiable Curves

Definition 44 (Regular, Dt.: regulär)

A $C^r$-curve $\gamma : I \rightarrow \mathbb{R}^n$ is called regular of order $k$, for some $0 < k \leq r$, if the vectors $\{\gamma'(t), \gamma''(t), \ldots, \gamma^{(k)}(t)\}$ are linearly independent for every $t \in I$. In particular, $\gamma$ is called regular if $\gamma'(t) \neq 0 \in \mathbb{R}^n$ for every $t \in I$.

Definition 45 (Singular, Dt.: singulär)

For a $C^1$-curve $\gamma : I \rightarrow \mathbb{R}^n$ and $t_0 \in I$, the point $\gamma(t_0)$ is called a singular point of $\gamma$ if $\gamma'(t_0) = 0$.

► Note: Regularity and singularity depend on the parametrization!
Equivalence of Parametrizations in $\mathbb{R}^n$

- Note that parametrizations of a curve (regarded as a set $C \subset \mathbb{R}^n$) need not be unique: Two different parametrizations $\gamma: I \rightarrow \mathbb{R}^n$ and $\beta: J \rightarrow \mathbb{R}^n$ may exist such that $C = \gamma(I) = \beta(J)$.

$$\gamma(t) := \begin{pmatrix} \cos 2\pi t \\ \sin 2\pi t \end{pmatrix} \quad \beta(t) := \begin{pmatrix} \cos 2\pi t \\ -\sin 2\pi t \end{pmatrix}$$

**Figure:** $\gamma(t)$ for $t \in [0, 0.9]$

**Figure:** $\beta(t)$ for $t \in [0, 0.9]$
Equivalence of Parametrizations in $\mathbb{R}^n$

**Definition 46 (Reparametrization, Dt.: Umparameterisierung)**

Let $\gamma : I \to \mathbb{R}^n$ and $\beta : J \to \mathbb{R}^n$ both be $C^r$-curves, for some $r \in \mathbb{N}_0$. We consider $\gamma$ and $\beta$ as *equivalent* if a function $\phi : I \to J$ exists, such that
\[
\beta(\phi(t)) = \gamma(t) \quad \forall t \in I,
\]
and

1. $\phi$ is continuous, strictly monotonously increasing and bijective,
2. both $\phi$ and $\phi^{-1}$ are $r$ times continuously differentiable.

In this case the parametric curve $\beta$ is called a *reparametrization* of $\gamma$.

**Caveat**

There is no universally accepted definition of a reparametrization! Some authors drop the monotonicity or the differentiability of $\phi$, while others even require $\phi$ to be smooth.
Arc Length

Definition 47 (Decomposition, Dt.: Unterteilung)

Consider $\gamma : I \rightarrow \mathbb{R}^n$, with $I := [a, b]$. A decomposition, $P$, of the closed interval $I$ is a sequence of $m + 1$ numbers $t_0, t_1, t_2, \ldots, t_m$, for some $m \in \mathbb{N}$, such that

$$a = t_0 < t_1 < t_2 < \cdots < t_m = b.$$

The length $L_P(\gamma)$ of the polygonal chain $(\gamma(t_0), \gamma(t_1), \gamma(t_2), \ldots, \gamma(t_m))$ that corresponds to the decomposition $t_0, t_1, t_2, \ldots, t_m$ is given by

$$L_P(\gamma) := \sum_{j=0}^{m-1} \| \gamma(t_{j+1}) - \gamma(t_j) \|

= \| \gamma(t_1) - \gamma(t_0) \| + \| \gamma(t_2) - \gamma(t_1) \| + \cdots + \| \gamma(t_m) - \gamma(t_{m-1}) \|.$$ 

We denote the set of all decompositions of $[a, b]$ by $\mathcal{P}[a, b]$. 

**Arc Length**

**Definition 48 (Arc length, Dt.: Bogenlänge)**

Consider \( \gamma : I \to \mathbb{R}^n \), with \( I := [a, b] \). The **arc length** of \( \gamma(I) \) is given by

\[
\sup \{ L_P(\gamma) : P \in \mathcal{P}[a, b] \},
\]

i.e., by the supremum (over all decompositions \( t_0, t_1, t_2, \ldots, t_m \) of \( I \)) of the length of the polygonal chain defined by \( \gamma(t_0), \gamma(t_1), \gamma(t_2), \ldots, \gamma(t_m) \).

**Definition 49 (Rectifiable, Dt.: rektifizierbar)**

If the arc length of \( \gamma : I \to \mathbb{R}^n \) is a finite number then \( \gamma(I) \) is called **rectifiable**.

**Lemma 50**

The arc length of a curve does not change for equivalent parametrizations.

**Sketch of Proof**: Suppose that \( \gamma(t) = \beta(\phi(t)) \) for all \( t \in I \), for \( \beta : J \to \mathbb{R}^n \). Every decomposition \( t_0, t_1, t_2, \ldots, t_m \) of \( I \) maps to a decomposition \( \phi(t_0), \phi(t_1), \phi(t_2), \ldots, \phi(t_m) \) of \( J \) such that \( \gamma(t_i) = \beta(\phi(t_i)) \) for all \( 1 \leq i \leq m \). Hence, there is a bijection from the set of decompositions of \( I \) to the set of decompositions of \( J \), and it does not matter which set is used for determining the supremum of all possible chain lengths. \( \square \)
Curves exist that are non-rectifiable, i.e., for which there is no upper bound on the length of their polygonal approximations.

Example of a non-rectifiable curve: The graph of the function defined by $f(0) := 0$ and $f(x) := x \sin \left( \frac{1}{x} \right)$ for $0 < x \leq a$, for some $a \in \mathbb{R}^+$. It defines a curve $\gamma(t) := \begin{pmatrix} t \\ f(t) \end{pmatrix}$.

The graph of $f(x) := x \sin \left( \frac{1}{x} \right)$ for $x \in [0, 2]$
Curves exist that are non-rectifiable, i.e., for which there is no upper bound on the length of their polygonal approximations.

Example of a non-rectifiable curve: The graph of the function defined by $f(0) := 0$ and $f(x) := x \sin \left(\frac{1}{x}\right)$ for $0 < x \leq a$, for some $a \in \mathbb{R}^+$. It defines a curve $\gamma(t) := \begin{pmatrix} t \\ f(t) \end{pmatrix}$.

The graph of $f(x) := x \sin \left(\frac{1}{x}\right)$ for $x \in [0, \frac{1}{2}]$
Arc Length: Non-Rectifiable Curve

- Curves exist that are non-rectifiable, i.e., for which there is no upper bound on the length of their polygonal approximations.
- Example of a non-rectifiable curve: The graph of the function defined by \( f(0) := 0 \) and \( f(x) := x \sin \left( \frac{1}{x} \right) \) for \( 0 < x \leq a \), for some \( a \in \mathbb{R}^+ \). It defines a curve \( \gamma(t) := \left( \begin{array}{c} t \\ f(t) \end{array} \right) \).

The graph of \( f(x) := x \sin \left( \frac{1}{x} \right) \) for \( x \in [0, \frac{1}{32}] \)
Example of a non-rectifiable closed curve: The *Koch snowflake* [Koch 1904].

The length of the curve after the $n$-th iteration is $(\frac{4}{3})^n$ times the original triangle perimeter. (Its fractal dimension is $\log_{\frac{4}{3}} \approx 1.261$.)
Arc Length

Lemma 51
If \( \gamma: I \to \mathbb{R}^n \) is a \( C^1 \)-curve then \( \gamma(I) \) is rectifiable.

Theorem 52
Let \( \gamma: I \to \mathbb{R}^n \) be a \( C^1 \) curve, with \( I := [a, b] \). Then the arc length of \( \gamma(I) \) is given by
\[
\int_a^b \| \gamma'(t) \| \, dt.
\]

Corollary 53
Let \( \gamma: I \to \mathbb{R}^n \) be a \( C^1 \) curve, and \( [a, b] \subseteq I \). Then the arc length of \( \gamma([a, b]) \) is given by
\[
\int_a^b \| \gamma'(t) \| \, dt.
\]
Definition 54 (*Speed, Dt.: Geschwindigkeit*)

If $\gamma: I \to \mathbb{R}^n$ is a $C^1$-curve then the vector $\gamma'(t)$ is the velocity vector at parameter $t$, and $\|\gamma'(t)\|$ gives the speed at parameter $t$, for all $t \in I$.

Definition 55 (*Natural parametrization*)

A $C^1$-curve $\gamma: I \to \mathbb{R}^n$ is called natural (or at unit speed) if $\|\gamma'(t)\| = 1$ for all $t \in I$.

Theorem 56

If $\gamma: I \to \mathbb{R}^n$, with $I := [a, b]$, is a regular curve then there exists an equivalent reparametrization $\tilde{\gamma}$ that has unit speed.
If \( \gamma(t_0) \) is a fixed point on the curve \( \gamma \), and \( \gamma(t_1) \), with \( t_1 > t_0 \), is another point, then the vector from \( \gamma(t_0) \) to \( \gamma(t_1) \) approaches the tangent vector to \( \gamma \) at \( \gamma(t_0) \) as the distance between \( t_1 \) and \( t_0 \) is decreased.

The infinite line through \( \gamma(t_0) \) that is parallel to this vector is known as the tangent line to the curve \( \gamma \) at point \( \gamma(t_0) \).

If we disregard the orientation of the tangent vector then we would like to obtain the same result for the tangent line by considering a point \( \gamma(t_1) \) with \( t_1 < t_0 \).
Tangent Vector

Definition 57 (Tangent vector)

Let $\gamma : I \to \mathbb{R}^n$ be a $C^1$-curve. If $\gamma'(t) \neq 0$ for $t \in I$ then $\gamma'(t)$ forms the tangent vector at the point $\gamma(t)$ of $\gamma$.

- The tangent vector indicates the forward direction of $\gamma$ relative to increasing parameter values.
- If $\gamma$ is at unit speed then $\gamma'(t)$ forms a unit vector.
- A parametrization of the tangent line $\ell$ that passes through $\gamma(t)$ is given by
  \[
  \ell(\lambda) = \gamma(t) + \lambda \gamma'(t) \quad \text{with} \ \lambda \in \mathbb{R}.
  \]
- If
  \[
  \gamma(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}
  \]
  is a curve in $\mathbb{R}^2$ then the vector
  \[
  \begin{pmatrix} -y'(t) \\ x'(t) \end{pmatrix}
  \]
  is normal on the tangent line at $\gamma(t)$. 

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Frenet Frame for Curves in $\mathbb{R}^3$

Definition 58 (*Frenet frame, Dt.: begleitendes Dreibein*)

Let $\gamma: I \to \mathbb{R}^3$ be a $C^2$ curve that is regular of order two. Then the *Frenet frame* (aka *moving trihedron*) at $\gamma(t)$ is defined as an orthonormal basis of vectors $T(t), N(t), B(t)$ as follows:

- $T(t) := \frac{\gamma'(t)}{\|\gamma'(t)\|}$ \; \text{unit tangent;}
- $N(t) := \frac{T'(t)}{\|T'(t)\|}$ \; \text{unit (principal) normal;}
- $B(t) := T(t) \times N(t)$ \; \text{unit binormal.}$

Lemma 59

Let $\gamma: I \to \mathbb{R}^3$ be a $C^2$ curve that is regular of order two, and define $T, N, B$ as in Def. 58. We get for all $t \in I$:

- $N(t)$ is normal to $T(t)$, and
- $B(t)$ is a unit vector.
Osculating Plane for a Curve in $\mathbb{R}^3$

Let $\gamma(t_0)$ be a fixed point on $\gamma$, and two other points $\gamma(t_1)$ and $\gamma(t_2)$ that move along $\gamma$.

Obviously, $\gamma(t_0)$, $\gamma(t_1)$ and $\gamma(t_2)$ define a plane.

As both $\gamma(t_1)$ and $\gamma(t_2)$ approach $\gamma(t_0)$, the plane determined by them approaches a limiting position.

This limiting plane is known as the osculating plane to the curve $\gamma$ at point $\gamma(t_0)$.

**Definition 60 (Osculating plane, Dt.: Schmiegeebene)**

Let $\gamma : I \to \mathbb{R}^3$ be a $C^2$ curve that is regular of order two. The osculating plane at the point $\gamma(t)$ is the plane spanned by $T(t)$ and $N(t)$, as defined in Def. 58.

Of course, the osculating plane contains the tangent line to $\gamma$ at $\gamma(t_0)$.

The osculating plane at the point $\gamma(t_0)$ contains also $\gamma''(t_0)$.

Thus, the binormal is the normal vector of the osculating plane.
Curvature of Curves in $\mathbb{R}^3$

- The curvature at a given point of a curve (in $\mathbb{R}^3$) is a measure of how quickly the curve changes direction at that point relative to the speed of the curve.

Definition 61 (*Curvature, Dt.: Krümmung*)

Let $\gamma : I \to \mathbb{R}^3$ be a $C^2$ curve that is regular. The *curvature* $\kappa(t)$ of $\gamma$ at the point $\gamma(t)$ is defined as

$$\kappa(t) := \frac{\|\gamma'(t) \times \gamma''(t)\|}{\|\gamma'(t)\|^3}.$$ 

Definition 62 (*Radius of curvature, Dt.: Krümmungsradius*)

Let $\gamma : I \to \mathbb{R}^3$ be a $C^2$ curve that is regular. If $\kappa(t) > 0$ then the *radius of curvature* $\rho(t)$ at the point $\gamma(t)$ is defined as

$$\rho(t) := \frac{1}{\kappa(t)}.$$
Consider the circle that
1. passes through $\gamma(t_0)$,
2. touches the tangent at $\gamma(t_0)$, and that
3. passes through another point $\gamma(t_1)$.

Now imagine that the difference between $t_1$ and $t_0$ is decreased.

In the limit, for $t_1 = t_0$, we get the so-called circle of curvature.

One can prove that
- the circle of curvature (aka “osculating circle”) lies in the osculating plane of $\gamma(t_0)$,
- its radius, the radius of curvature, is given by $\rho(t_0)$, and that
- its center lies on the ray with direction vector $N(t_0)$ that starts at $\gamma(t_0)$. 
Curvature of Curves in $\mathbb{R}^3$: Inflection

Definition 63 (Point of inflection, Dt.: Wendepunkt)

Let $\gamma: I \rightarrow \mathbb{R}^3$ be a $C^2$-curve that is regular. If for all $t \in I$ the second derivative $\gamma''$ does not vanish, i.e., if $\gamma''(t) \neq 0$, then a point $\gamma(t)$ for which $\kappa(t) = 0$ holds is called a point of inflection of $\gamma$.

Lemma 64

Let $\gamma: I \rightarrow \mathbb{R}^3$ be a $C^2$-curve that is regular such that for all $t \in I$ the second derivative $\gamma''$ does not vanish. Then $\gamma(t)$ is a point of inflection of $\gamma$ if and only if $\gamma'(t)$ and $\gamma''(t)$ are collinear.

Hence, at a point of inflection the radius of curvature is infinite and the circle of curvature degenerates to the tangent.
Curvature of Curves in $\mathbb{R}^3$

**Lemma 65**

Let $\gamma: I \rightarrow \mathbb{R}^3$ be a $C^3$-curve at unit speed that is regular of order two. Then the following simplified formula holds:

$$\kappa(t) = \|\gamma''(t)\|$$

**Sketch of Proof:** Recall that, in general,

$$\kappa(t) = \frac{\|\gamma'(t) \times \gamma''(t)\|}{\|\gamma'(t)\|^3}.$$
Curvature of Curves in $\mathbb{R}^2$

Lemma 66

Let $\gamma: I \to \mathbb{R}^2$ be a $C^2$-curve that is regular, with $\gamma(t) = (x(t), y(t))$. Then $\kappa(t)$ of $\gamma$ at the point $\gamma(t)$ is given as

$$\kappa(t) = \frac{|x'(t)y''(t) - x''(t)y'(t)|}{\left(\left(x'(t)\right)^2 + \left(y'(t)\right)^2\right)^{3/2}}.$$

Sketch of Proof: Recall that, in general,

$$\kappa(t) = \frac{\|\gamma'(t) \times \gamma''(t)\|}{\|\gamma'(t)\|^3}.$$

Corollary 67

Let $\gamma: I \to \mathbb{R}^2$ be a $C^2$-curve that is regular, with $\gamma(t) = (t, y(t))$. Then $\kappa(t)$ of $\gamma$ at the point $\gamma(t)$ is given as

$$\kappa(t) = \frac{|y''(t)|}{\left(1 + \left(y'(t)\right)^2\right)^{3/2}}.$$
Consider two curves $\beta: [a, b] \to \mathbb{R}^n$ and $\gamma: [c, d] \to \mathbb{R}^n$.

Suppose that $\beta(b) = \gamma(c) =: p$.

We are interested in checking how “smoothly” $\beta$ and $\gamma$ join at the joint $p$.

Definition 68 ($C^k$-continuous at joint, Dt.: $C^k$-stetiger Übergang)

Let $\beta: [a, b] \to \mathbb{R}^n$ and $\gamma: [c, d] \to \mathbb{R}^n$ be $C^k$-curves. If

$$\beta^{(i)}(b) = \gamma^{(i)}(c) \quad \text{for all } i \in \{0, \ldots, k\}$$

then $\beta$ and $\gamma$ are $C^k$-continuous at joint $p := \beta(b)$.

Of course, one-sided derivatives are to be considered in Def. 68.
Parametric Continuity of a Curve

- $C^0$-continuity implies that the end point of one curve is the start point of the second curve, i.e., they share a common joint.
- $C^1$-continuity implies that the speed does not change at $p$.
- $C^2$-continuity implies that the acceleration does not change at $p$.
- Parametric continuity is important for animations: If an object moves along $\beta$ with constant parametric speed, then there should be no sudden jump once it moves along $\gamma$.

Definition 69 (Curvature continuous, Dt.: krümmungsstetig)

Let $\beta : [a, b] \rightarrow \mathbb{R}^3$ and $\gamma : [c, d] \rightarrow \mathbb{R}^3$ be $C^2$-curves, with $\beta(b) = \gamma(c) =: p$. If the curvatures of $\beta$ and $\gamma$ are equal at $p$ then $\beta$ and $\gamma$ are said to be curvature continuous at $p$.

Caveat

$C^1$-continuity plus curvature continuity need not imply $C^2$-continuity!

- Unfortunately, this important fact is missed frequently, and curvature continuity is often (wrongly) taken as a synonym for $C^2$-continuity . . .
Problems with Parametric Continuity

- Note that parametric continuity depends on the particular parametrizations of $\beta$ and $\gamma$.

- Consider three collinear points $p$, $q$, and $r$ which define two straight-line segments joined at their common endpoint $q$:

  $\beta(t) := p + t(q - p), \quad t \in [0, 1]$
  $\gamma(t) := q + (t - 1)(r - q), \quad t \in [1, 2]$

- Of course, $\beta$ and $\gamma$ are $C^0$-continuous at $q$.

- However, $\beta'(1) = q - p$ while $\gamma'(1) = r - q$. Thus, in general, $\beta$ and $\gamma$ will not be $C^1$-continuous at $q$.

- $C^1$-continuity at $q$ could be achieved by resorting to arc-length parametrizations for $\beta$ and $\gamma$:

  $\beta(t) := p + \frac{t}{\|q - p\|}(q - p), \quad t \in [0, \|q - p\|]$  
  $\gamma(t) := q + \frac{t - \|q - p\|}{\|r - q\|}(r - q), \quad t \in [\|q - p\|, \|q - p\| + \|r - q\|]$
Geometric Continuity

- $G^0$-continuity equals $C^0$-continuity: The curves $\beta$ and $\gamma$ share a common joint $p$.

Definition 70 ($G^1$-continuous at joint, Dt.: $G^1$-stetiger Übergang)

Let $\beta : [a, b] \rightarrow \mathbb{R}^n$ and $\gamma : [c, d] \rightarrow \mathbb{R}^n$ be $C^1$-curves, with $\beta(b) = \gamma(c) = p$. If

$$0 \neq \beta'(b) = \lambda \cdot \gamma'(c)$$

for some $\lambda \in \mathbb{R}^+$

then $\beta$ and $\gamma$ are $G^1$-continuous at joint $p$.

- $G^1$-continuity means that $\beta$ and $\gamma$ share the tangent line at $p$.
- Higher-order geometric continuities are a bit tricky to define formally for $k \geq 2$.
- $G^2$-continuity means that $\beta$ and $\gamma$ share the tangent line and also the same center of curvature at $p$.
- In general, $G^k$-continuity exists at $p$ if $\beta$ and $\gamma$ can be reparameterized such that they join with $C^k$-continuity at $p$.
- $C^k$-continuity implies $G^k$-continuity.
- Note: Reflections on a surface (e.g., a car body) will not appear smooth unless $G^2$-continuity is achieved.
Parametric Surface in $\mathbb{R}^3$

Definition 71 (*Parametric surface*)

Let $\Omega \subseteq \mathbb{R}^2$. A continuous mapping $\alpha: \Omega \rightarrow \mathbb{R}^3$ is called a *parametrization* of $\alpha(\Omega)$, and $\alpha(\Omega)$ is called the (parametric) *surface* parameterized by $\alpha$.

- For instance, every point on the surface of Earth can be described by the geographic coordinates longitude and latitude.
- Note that parametrizations of a surface (regarded as a set $S \subset \mathbb{R}^3$) need not be unique: two different parametrizations $\alpha$ and $\beta$ may exist such that $S = \alpha(\Omega_1) = \beta(\Omega_2)$.
- For simplicity, we will not distinguish between a surface and one of its parametrizations if the meaning is clear.

![Diagram](image-url)
Sample Parametric Surface: Frustum of a Paraboloid

\[ \alpha : [0, 1] \times [0, 2\pi] \to \mathbb{R}^3 \]

\[ \alpha(u, v) := \begin{pmatrix} u \cos v \\ u \sin v \\ 2u^2 \end{pmatrix} \]
Sample Parametric Surface: Torus

\[ \alpha : [0, 2\pi]^2 \rightarrow \mathbb{R}^3 \]

\[ \alpha(u, v) := \begin{pmatrix} (2 + \cos v) \cos u \\ (2 + \cos v) \sin u \\ \sin v \end{pmatrix} \]
Definition 72 (Regular parametrization, Dt.: reguläre (od. zulässige) Param.)

Let $\Omega \subseteq \mathbb{R}^2$. A continuous mapping $\alpha : \Omega \rightarrow \mathbb{R}^3$ in the variables $u$ and $v$ is called a regular parametrization of $\alpha(\Omega)$ if

1. $\alpha$ is differentiable on $\Omega$,
2. $\frac{\partial \alpha}{\partial u}(u_0, v_0)$ and $\frac{\partial \alpha}{\partial v}(u_0, v_0)$ are linearly independent for all $(u_0, v_0)$ in $\Omega$.

Note that $\frac{\partial \alpha}{\partial u}(u_0, v_0)$ and $\frac{\partial \alpha}{\partial v}(u_0, v_0)$ are linearly independent if and only if

$$\frac{\partial \alpha}{\partial u}(u_0, v_0) \times \frac{\partial \alpha}{\partial v}(u_0, v_0) \neq 0.$$ 

Definition 73 (Singular point, Dt.: singulärer Punkt)

Let $\Omega \subseteq \mathbb{R}^2$. A point $(u_0, v_0) \in \Omega$ is a singular point of a differentiable parametrization $\alpha : \Omega \rightarrow \mathbb{R}^3$ if $\frac{\partial \alpha}{\partial u}(u_0, v_0)$ and $\frac{\partial \alpha}{\partial v}(u_0, v_0)$ are linearly dependent.


Tangent Plane and Normal Vector

- Recall that the tangent vector at a point $\gamma(t)$ of a parametric curve is given by $\gamma'(t)$.
- If $\gamma$ lies on a surface $\alpha(\Omega)$, then $\gamma'(t)$ can be seen as some form of partial derivative of $\alpha$. This motivates the following definition.

Definition 74 (Tangent plane, Dt.: Tangentialebene)

Consider a regular parametrization $\alpha : \Omega \to \mathbb{R}^3$ of a surface $S$. For $(u, v) \in \Omega$, the tangent plane $\varepsilon(u, v)$ of $S$ at $\alpha(u, v)$ is the plane through $\alpha(u, v)$ that is spanned by the vectors

$$\frac{\partial \alpha}{\partial u}(u, v) \quad \text{and} \quad \frac{\partial \alpha}{\partial v}(u, v).$$

Definition 75 (Normal vector, Dt.: Normalvektor)

Consider a regular parametrization $\alpha : \Omega \to \mathbb{R}^3$ of a surface $S$. For $(u, v) \in \Omega$, the normal vector $N(u, v)$ of $S$ at $\alpha(u, v)$ is given by

$$N(u, v) := \frac{\partial \alpha}{\partial u}(u, v) \times \frac{\partial \alpha}{\partial v}(u, v).$$
Bézier Curves and Surfaces
Bernstein Basis Polynomials
Bézier Curves
Bézier Surfaces
Bernstein Basis Polynomials

Definition 76 (Bernstein basis polynomials)

The \( n + 1 \) Bernstein basis polynomials of degree \( n \), for \( n \in \mathbb{N}_0 \), are defined as

\[
B_{k,n}(x) := \binom{n}{k} x^k (1 - x)^{n-k} \quad \text{for} \ k \in \{0, 1, \ldots, n\}.
\]

- We use the convention \( 0^0 := 1 \).
- For convenience purposes, we define \( B_{k,n}(x) := 0 \) for \( k < 0 \) or \( k > n \).
- \( B_{0,0}(x) = 1 \).
- \( B_{0,1}(x) = 1 - x \) and \( B_{1,1}(x) = x \).
- \( B_{0,2}(x) = (1 - x)^2 \) and \( B_{1,2}(x) = 2x(1 - x) \) and \( B_{2,2}(x) = x^2 \).

- Introduced by Sergei N. Bernstein in 1911 for a constructive proof of Weierstrass’ Approximation Theorem 177.
Bernstein Basis Polynomials

- All Bernstein basis polynomials of degree $n = 0$ over the interval $[0, 1]$:

$$1$$
Bernstein Basis Polynomials

All Bernstein basis polynomials of degree $n = 1$ over the interval $[0, 1]$: $1 - x$  $x$
Bernstein Basis Polynomials

All Bernstein basis polynomials of degree $n = 2$ over the interval $[0, 1]$:

\[
(1 - x)^2 \quad 2x(1 - x) \quad x^2
\]
Bernstein Basis Polynomials

All Bernstein basis polynomials of degree \( n = 3 \) over the interval \([0, 1]\):

\[
\begin{align*}
(1 - x)^3 & \quad 3x(1 - x)^2 & \quad 3x^2(1 - x) & \quad x^3
\end{align*}
\]
 Bernstein Basis Polynomials

All Bernstein basis polynomials of degree $n = 4$ over the interval $[0, 1]$: 

$$(1 - x)^4 \quad 4x(1 - x)^3 \quad 6x^2(1 - x)^2 \quad 4x^3(1 - x) \quad x^4$$
Bernstein Basis Polynomials

- All Bernstein basis polynomials of degree $n = 5$ over the interval $[0, 1]$:

$$(1 - x)^5 \quad 5x(1 - x)^4 \quad 10x^2(1 - x)^3 \quad 10x^3(1 - x)^2 \quad 5x^4(1 - x) \quad x^5$$
Recursion Formula for Bernstein Basis Polynomials

**Lemma 77**

For all \( n \in \mathbb{N} \) and \( k \in \mathbb{N}_0 \) with \( k \leq n \), the Bernstein basis polynomial \( B_{k,n}(x) \) of degree \( n \) can be written as the sum of two basis polynomials of degree \( n - 1 \):

\[
B_{k,n}(x) = x \cdot B_{k-1,n-1}(x) + (1 - x) \cdot B_{k,n-1}(x)
\]

**Proof:** Let \( n \in \mathbb{N} \) and \( k \in \mathbb{N}_0 \) with \( k \leq n \) be arbitrary but fixed, and recall that

\[
B_{k,n}(x) \overset{\text{Def. 76}}{=} \binom{n}{k} x^k (1 - x)^{n-k} \quad \text{and} \quad \binom{n}{k} \overset{\text{Thm. 4}}{=} \left( \binom{n-1}{k-1} + \binom{n-1}{k} \right).
\]

We get

\[
B_{k,n}(x) = \binom{n-1}{k-1} x^k (1 - x)^{n-k} + \binom{n-1}{k} x^k (1 - x)^{n-k}
\]

\[
= x \left( \binom{n-1}{k-1} x^{k-1} (1 - x)^{(n-1)-(k-1)} + (1 - x) \binom{n-1}{k} x^k (1 - x)^{(n-1)-k} \right)
\]

\[
= x \cdot B_{k-1,n-1}(x) + (1 - x) \cdot B_{k,n-1}(x).
\]

\( \square \)
Properties of Bernstein Basis Polynomials

Lemma 78

For all $n \in \mathbb{N}$ and $k \in \mathbb{N}_0$ with $k \leq n$, the Bernstein basis polynomial $B_{k,n-1}$ can be written as linear combination of Bernstein basis polynomials of degree $n$:

\[ B_{k,n-1}(x) = \frac{n-k}{n} B_{k,n}(x) + \frac{k+1}{n} B_{k+1,n}(x). \]

Lemma 79

For all $n, k \in \mathbb{N}_0$ with $k \leq n$, the Bernstein basis polynomial $B_{k,n}$ is non-negative over the unit interval:

\[ B_{k,n}(x) \geq 0 \quad \text{for all } x \in [0, 1]. \]

Proof: Recall the definition of the Bernstein basis polynomials:

\[ B_{k,n}(x) \overset{\text{Def. 76}}{=} \binom{n}{k} (x)^k (1-x)^{n-k} \geq 0 \quad \text{for all } x \in [0, 1]. \]
Lemma 80 \textit{(Partition of unity, Dt.: Zerlegung der Eins)}

For all \( n \in \mathbb{N}_0 \), the \( n + 1 \) Bernstein basis polynomials of degree \( n \) form a partition of unity, i.e., they sum up to one:

\[
\sum_{k=0}^{n} B_{k,n}(x) = 1 \quad \text{for all } x \in [0, 1].
\]

\textbf{Proof}: Trivial for \( n = 0 \). Now recall the Binomial Theorem 5, for \( a, b \in \mathbb{R} \) and \( n \in \mathbb{N} \):

\[
(a + b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}
\]

Then the claim is an immediate consequence by setting \( a := x \) and \( b := 1 - x \):

\[
1 = (x + (1 - x))^n = \sum_{k=0}^{n} \binom{n}{k} x^k (1 - x)^{n-k} = \sum_{k=0}^{n} B_{k,n}(x).
\]
Properties of Bernstein Basis Polynomials

Lemma 81

For all $n \in \mathbb{N}_0$ and any set of $n + 1$ points in $\mathbb{R}^2$ with position vectors $p_0, p_1, p_2, \ldots, p_n$, the term

$$p_0 B_{0,n}(t) + p_1 B_{1,n}(t) + \cdots + p_n B_{n,n}(t)$$

forms a convex combination of these points for all $t \in [0, 1]$.

Proof: This is an immediate consequence of Lem. 79 and Lem. 80.

Corollary 82 (Convex hull property)

For all $n \in \mathbb{N}_0$ and any set of $n + 1$ points in $\mathbb{R}^2$ with position vectors $p_0, p_1, p_2, \ldots, p_n$, the point

$$p_0 B_{0,n}(t) + p_1 B_{1,n}(t) + \cdots + p_n B_{n,n}(t)$$

lies within $CH(\{p_0, p_1, p_2, \ldots, p_n\})$ for all $t \in [0, 1]$.

Proof: Recall that $CH(\{p_0, p_1, p_2, \ldots, p_n\})$ equals the set of all convex combinations of $p_0, p_1, p_2, \ldots, p_n$. 

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Derivatives of Bernstein Basis Polynomials

Lemma 83
For $n, k \in \mathbb{N}_0$ and $i \in \mathbb{N}$ with $i \leq n$, the $i$-th derivative of $B_{k,n}(x)$ can be written as a linear combination of Bernstein basis polynomials of degree $n - i$:

$$B_{k,n}^{(i)}(x) = \frac{n!}{(n-i)!} \sum_{j=0}^{i} (-1)^{i-j} \binom{i}{j} B_{k-j,n-i}(x)$$

Corollary 84
For $n, k \in \mathbb{N}_0$, the first and second derivative of $B_{k,n}(x)$ are given as follows:

$$B_{k,n}'(x) = n(B_{k-1,n-1}(x) - B_{k,n-1}(x))$$

$$B_{k,n}''(x) = n(n-1)(B_{k-2,n-2}(x) - 2B_{k-1,n-2}(x) + B_{k,n-2}(x))$$
Bernstein Basis Polynomials Form a Basis

Lemma 85

The \( n + 1 \) Bernstein basis polynomials \( B_{0,n}, B_{1,n}, \ldots, B_{n,n} \) are linearly independent, for all \( n \in \mathbb{N}_0 \).

Proof: We do a proof by induction.

I.B.: The claim is obviously true for \( n = 0 \) and \( n = 1 \).

I.H.: Suppose that the claim is true for an arbitrary but fixed \( n - 1 \in \mathbb{N}_0 \), i.e., that

\[
\sum_{k=0}^{n-1} \lambda_k B_{k,n-1}(x) = 0
\]

implies \( \lambda_0 = \lambda_1 = \ldots = \lambda_{n-1} = 0 \).

I.S.: Suppose that \( \sum_{k=0}^{n} \lambda_k B_{k,n}(x) = 0 \) for some \( \lambda_0, \lambda_1, \ldots, \lambda_n \in \mathbb{R} \). Then we get

\[
0 = \sum_{k=0}^{n} \lambda_k B'_{k,n}(x) \overset{\text{Lem. 83}}{=} \sum_{k=0}^{n} \lambda_k \cdot n \cdot (B_{k-1,n-1}(x) - B_{k,n-1}(x))
\]

\[
= n \left( \sum_{k=0}^{n-1} \lambda_{k+1} B_{k,n-1}(x) - \sum_{k=0}^{n-1} \lambda_k B_{k,n-1}(x) \right)
\]

\[
= n \sum_{k=0}^{n-1} \mu_k B_{k,n-1}(x) \quad \text{with } \mu_k := \lambda_{k+1} - \lambda_k \text{ for } 0 \leq k \leq n - 1.
\]

The I.H. implies \( \mu_0 = \mu_1 = \cdots = \mu_{n-1} = 0 \) and, thus, \( \lambda_0 = \lambda_1 = \cdots = \lambda_n \), which implies \( \lambda_0 = \lambda_1 = \cdots = \lambda_n = 0 \). (Recall Partition of Unity, Lem. 80.)

\[\square\]
Bernstein Basis Polynomials Form a Basis

Lemma 86
For all $n, i \in \mathbb{N}_0$, with $i \leq n$, we have

$$x^i = \sum_{k=i}^{n} \binom{k}{i} B_{k,n}(x).$$

Theorem 87
The Bernstein basis polynomials of degree $n$ form a basis of the vector space of polynomials of degree up to $n$ over $\mathbb{R}$, for all $n \in \mathbb{N}_0$.

\textbf{Proof}: This is an immediate consequence of either Lem. 85 or Lem. 86.
Bézier Curves

- Discovered in the late 1950s by Paul de Casteljau at Citroën and in the early 1960s by Pierre E. Bézier at Renault, and first published by Bézier in 1962. (Citroën allowed de Casteljau to publish his results in 1974 for the first time.)

- The idea is to specify a curve by using points which control its shape: control points. The figure shows a Bézier curve of degree 10 with 11 control points.

- Bézier curves formed the foundations of Citroën’s UNISURF CAD/CAM system.

- TrueType fonts use font descriptions made of composite quadratic Bézier curves; PostScript, METAFONT, and SVG use composite Béziers made of cubic Bézier curves.
Bézier Curves

Definition 88 (Bézier curve)

Suppose that we are given \( n + 1 \) control points with position vectors \( p_0, p_1, \ldots, p_n \) in the plane \( \mathbb{R}^2 \), for \( n \in \mathbb{N} \). The Bézier curve \( B : [0, 1] \to \mathbb{R}^2 \) defined by \( p_0, p_1, \ldots, p_n \) is given by

\[
B(t) := \sum_{i=0}^{n} B_{i,n}(t) p_i \quad \text{for } t \in [0, 1],
\]

where \( B_{i,n}(t) := \binom{n}{i} t^i (1 - t)^{n-i} \) is the \( i \)-th Bernstein basis polynomial of degree \( n \).

- The weighted average of all control points gives a location on the curve relative to the parameter \( t \). The weights are given by the coefficients \( B_{i,n} \).
- The polygonal chain \( p_0, p_1, p_3, \ldots, p_{n-1}, p_n \) is called the control polygon, and its individual segments are referred to as legs.
- Although not explicitly required, it is generally assumed that the control points are distinct, except for possibly \( p_0 \) and \( p_n \) being identical.
- Of course, the same definition and the subsequent math can be applied to \( p_0, p_1, \ldots, p_n \in \mathbb{R}^d \) for some \( d \in \mathbb{N} \) with \( d > 2 \).
Properties of Bézier Curves

Lemma 89

A Bézier curve defined by \( n + 1 \) control points is (coordinate-wise) a polynomial of degree \( n \).

Proof: It is the sum of \( n + 1 \) Bernstein basis polynomials of degree \( n \).

Lemma 90

A Bézier curve starts in the first control point and ends in the last control point.

Proof: Recall that

\[
B_{i,n}(0) = \begin{cases} 
1 & \text{for } i = 0, \\
0 & \text{for } i > 0.
\end{cases}
\]

Hence, \( \mathcal{B}(0) = B_{0,n}(0)p_0 = p_0 \). Similarly for \( B_{i,n}(1) \) and \( \mathcal{B}(1) \).
Properties of Bézier Curves

Lemma 91 (Convex hull property)

A Bézier curve lies completely inside the convex hull of its control points.

Proof: This is nothing but a re-formulation of Cor. 82. □

Lemma 92 (Variation diminishing property)

If a straight line intersects the control polygon of a Bézier curve \( k \) times then it intersects the actual Bézier curve at most \( k \) times.

Lemma 93 (Symmetry property)

The following identity holds for all \( n \in \mathbb{N} \), all \( p_0, \ldots, p_n \in \mathbb{R}^2 \) and all \( t \in [0,1] \):

\[
\sum_{i=0}^{n} B_{i,n}(t)p_i = \sum_{i=0}^{n} B_{i,n}(1 - t)p_{n-i}.
\]
Properties of Bézier Curves

Lemma 94 (Affine invariance)

Any Bézier representation is affinely invariant, i.e., given any affine map $\pi$, the image curve $\pi(B)$ of a Bézier curve $B: [0, 1] \rightarrow \mathbb{R}^2$ with control points $p_0, p_1, \ldots, p_n$ has the control points $\pi(p_0), \pi(p_1), \ldots, \pi(p_n)$ over $[0, 1]$.

Proof: Consider an affine map $\pi: \mathbb{R}^2 \rightarrow \mathbb{R}^2$. Hence, $\pi(x) = A \cdot x + v$, for some $2 \times 2$ matrix $A$, and $x, v \in \mathbb{R}^2$. We get

$$\pi(B(t)) = \pi \left( \sum_{i=0}^{n} B_{i,n}(t)p_i \right) = A \cdot \left( \sum_{i=0}^{n} B_{i,n}(t)p_i \right) + v$$

$$= \sum_{i=0}^{n} B_{i,n}(t)A \cdot p_i + \sum_{i=0}^{n} B_{i,n}(t)v = \sum_{i=0}^{n} B_{i,n}(t)(A \cdot p_i + v)$$

$$= \sum_{i=0}^{n} B_{i,n}(t)\pi(p_i).$$

$\square$
Suppose that we shift one control point \( p_j \) to a new location \( p_j + v \).

The corresponding Bézier curve \( \mathcal{B} \) is transformed to \( \mathcal{B}^* \) as follows:

\[
\mathcal{B}^*(t) = \left( \sum_{i=0}^{j-1} B_{i,n}(t) p_i \right) + B_{j,n}(t)(p_j + v) + \left( \sum_{i=j+1}^{n} B_{i,n}(t) p_i \right) = \\
\sum_{i=0}^{n} B_{i,n}(t) p_i + B_{j,n}(t)v = \mathcal{B}(t) + B_{j,n}(t)v
\]

Now recall that \( B_{j,n}(t) \neq 0 \) for all \( t \) with \( 0 < t < 1 \). Hence, a modification of just one control point results in a global change of the entire Bézier curve.
Evaluation of a Bézier Curve

- For $0 < t < 1$ we can locate a point $q$ on a line segment $\overline{pr}$ such that it divides the line segment into portions of relative length $t$ and $1 - t$, i.e., according to the ratio $t : (1 - t)$.

- Of course, $q$ is given by the linear interpolation

$$q = p + t(r - p) = (1 - t) \cdot p + t \cdot r.$$  

- Similarly, we can compute a point on a Bézier curve such that the curve is split into portions of relative length $t$ and $1 - t$.
  - On every leg $\overline{p_{j-1}p_j}$ of the control polygon we compute a point $p_{1j}$ which divides it according to the ratio $t : (1 - t)$.
  - In total we get $n$ new points which define a new polygonal chain with $n - 1$ legs.
  - This new polygonal chain can be used to construct another polygonal chain with $n - 2$ legs.
  - This process can be repeated $n$ times, i.e., until we are left with a single point.
  - It was proved by de Casteljau that this point corresponds to the point $B(t)$ sought.
De Casteljau’s Algorithm

- Sample run of de Casteljau’s algorithm for $t := \frac{1}{4}$.
- The points are indexed in the form $i, j$, where $i$ denotes the number of the iteration and $j + 1$ denotes the leg defined by the control points $p_{i,j}$ and $p_{i,j+1}$.
De Casteljau’s Algorithm

- Sample run of de Casteljau’s algorithm for $t := 1/4$.
- The points are indexed in the form $i, j$, where $i$ denotes the number of the iteration and $j + 1$ denotes the leg defined by the control points $p_{i,j}$ and $p_{i,j+1}$.

```
p0 =: p00
p1 =: p01
p2 =: p02
p3 =: p03
p4 =: p04
p5 =: p05
p0
p1
p2
p3
p4
p5
p2
p3
```
De Casteljau’s Algorithm

- Sample run of de Casteljau’s algorithm for $t := 1/4$.
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- Sample run of de Casteljau’s algorithm for \( t := 1/4 \).
- The points are indexed in the form \( i, j \), where \( i \) denotes the number of the iteration and \( j + 1 \) denotes the leg defined by the control points \( p_{i,j} \) and \( p_{i,j+1} \).
De Casteljau’s Algorithm

▶ Sample run of de Casteljau’s algorithm for \( t := \frac{1}{4} \).

▶ The points are indexed in the form \( i, j \), where \( i \) denotes the number of the iteration and \( j + 1 \) denotes the leg defined by the control points \( p_{i,j} \) and \( p_{i,j+1} \).

▶ The union of these legs \( p_{i,j}, p_{i,j+1} \) is known as de Casteljau net.
De Casteljau’s Algorithm

- Numerically very stable, since only convex combinations are used!

```c
/** Evaluates a Bezier curve at parameter t by applying de Casteljau’s algorithm
 * @param p: array of n+1 control points
 * @param n: the degree of the Bezier curve
 * @param t: the parameter
 * @return the evaluation result
 */

point DeCasteljau(point *p, int n, double t) {
    for (int i = 1; i <= n; ++i)
        for (int j = 0; j <= n-i; ++j)
            p[j] = (1-t) * p[j] + t * p[j+1];

    return p[0];
}
```

\[ p_0 =: p_{00} \quad p_1 =: p_{01} \quad p_2 =: p_{02} \quad p_3 =: p_{03} \quad p_4 =: p_{04} \quad p_5 =: p_{05} \]
\[ p_{10} \quad p_{11} \quad p_{12} \quad p_{13} \quad p_{14} \]
\[ p_{20} \quad p_{21} \quad p_{22} \quad p_{23} \]
\[ p_{30} \quad p_{31} \quad p_{32} \]
\[ p_{40} \quad p_{41} \]
\[ p_{50} \]
De Casteljau’s Algorithm: Correctness

- The point $p_{10}$ is obtained as
  \[ p_{10} = (1 - t) \cdot p_{00} + t \cdot p_{01}. \]

- Hence, the contribution of $p_{01}$ to $p_{10}$ is $t \cdot p_{01}$.

- Since $p_{20}$ is obtained as
  \[ p_{20} = (1 - t) \cdot p_{10} + t \cdot p_{11}, \]

  the contribution of $p_{01}$ to $p_{20}$ via $p_{10}$ is

  \[ (1 - t)p_{10} = t(1 - t) \cdot p_{01}. \]

- Similarly, the contribution of $p_{01}$ to $p_{20}$ via $p_{11}$ is

  \[ t(1 - t) \cdot p_{01}. \]
De Casteljau’s Algorithm: Correctness

▶ Each path from $p_{0i}$ to $p_{n0}$ is constrained to a diamond shape anchored at $p_{0i}$ and $p_{n0}$.

▶ An inductive argument shows that each path from $p_{0i}$ to $p_{n0}$ consists of $i$ north-east arrows, i.e., multiplications by $t$, and $n - i$ south-east arrows, i.e., multiplications by $(1 - t)$.

▶ Thus, the contribution of $p_{0i}$ to $p_{n0}$ is

$$t^i(1 - t)^{n-i} \cdot p_{0i},$$

along each path from $p_{0i}$ to $p_{n0}$. 

$p_0 =: p_{00} \rightarrow p_{10} \rightarrow p_{20} \rightarrow p_{30} \rightarrow p_{40} \rightarrow p_{50}$

$p_1 =: p_{01} \rightarrow p_{11} \rightarrow p_{21} \rightarrow p_{31} \rightarrow p_{41}$

$p_2 =: p_{02} \rightarrow p_{12} \rightarrow p_{22} \rightarrow p_{32}$

$p_3 =: p_{03} \rightarrow p_{13} \rightarrow p_{23}$

$p_4 =: p_{04} \rightarrow p_{14}$

$p_5 =: p_{05}$
De Casteljau’s Algorithm: Correctness

How many different paths exist from $p_{0i}$ to $p_{n0}$? This is equivalent to asking “how many different ways exist to place $i$ north-east arrows on a total of $n$ possible positions?”, and the answer is given by $\binom{n}{i}$.

Thus, the total contribution of $p_{0i}$ to $p_{n0}$, along all paths from $p_{0i}$ to $p_{n0}$, is

$$\binom{n}{i} \cdot t^i(1 - t)^{n-i} p_{0i}.$$

This is, however, precisely the weight of $p_{0i}$, i.e., $p_i$ in the definition of a Bézier curve (Def. 88).
Evaluation of a Bézier Curve Using Horner’s Scheme

- Horner’s scheme can also be used for evaluating a Bézier curve.

- After rewriting $B(t)$ as

$$B(t) = \sum_{i=0}^{n} B_{i,n}(t)p_i = \sum_{i=0}^{n} \binom{n}{i} t^i (1-t)^{n-i} p_i$$

$$= (1-t)^n \left( \sum_{i=0}^{n} \binom{n}{i} \left( \frac{t}{1-t} \right)^i p_i \right),$$

one evaluates the sum for the value $\frac{t}{1-t}$, and then multiplies by $(1-t)^n$.

- This method becomes unstable if $t$ is close to one. In this case, one can resort to Lem. 93, which gives the identity

$$B(t) = t^n \left( \sum_{i=0}^{n} \binom{n}{i} \left( \frac{1-t}{t} \right)^i p_{n-i} \right).$$

- In any case, Horner’s scheme tends to be faster but numerically more problematic than de Casteljau’s algorithm.

- [Woźny&Chudy (2019)] explain an algorithm that uses only convex combinations of the control points and consumes $O(n)$ time.
Bernstein Polynomials and Polar Forms

Theorem 95

Let \( n, d \in \mathbb{N} \). For every polynomial function \( F: \mathbb{R} \to \mathbb{R}^d \) of degree at most \( n \) there exists exactly one symmetric and multi-affine function \( f: \mathbb{R}^n \to \mathbb{R}^d \), which is called the polar form (aka “blossom”, Dt.: Polarform) of \( F \), such that

1. for all \( i \in \{1, 2, \ldots, n\} \), all \( x_1, x_2, \ldots, x_n \in \mathbb{R} \), all \( k \in \mathbb{N} \), all \( y_1, y_2, \ldots, y_k \in \mathbb{R} \) and all \( \alpha_1, \alpha_2, \ldots, \alpha_k \in \mathbb{R} \) with \( \sum_{j=1}^k \alpha_j = 1 \)

\[
f(x_1, \ldots, x_{i-1}, \sum_{j=1}^k \alpha_j y_j, x_{i+1}, \ldots, x_n) = \sum_{j=1}^k \alpha_j f(x_1, \ldots, x_{i-1}, y_j, x_{i+1}, \ldots, x_n)
\]

2. for all \( i, j \in \{1, 2, \ldots, n\} \)

\[
f(x_1, x_2, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_{j-1}, x_j, x_{j+1}, \ldots, x_n) = f(x_1, x_2, \ldots, x_{i-1}, x_j, x_{i+1}, \ldots, x_{j-1}, x_i, x_{j+1}, \ldots, x_n),
\]

3. for all \( x \in \mathbb{R} \)

\[
F(x) = f(x, x, \ldots, x), \text{ i.e., } F \text{ is the diagonal of } f.
\]
Bernstein Polynomials and Polar Forms

Lemma 96

Let $n \in \mathbb{N}$ and $a_0, a_1, \ldots, a_n \in \mathbb{R}$, and $F(x) := \sum_{i=0}^{n} a_i x^i$. Then $f : \mathbb{R}^n \rightarrow \mathbb{R}$ with

$$f(x_1, x_2, \ldots, x_n) := \sum_{i=0}^{n} a_i \frac{1}{\binom{n}{i}} \left( \sum_{\substack{I \subseteq \{1, \ldots, n\} \mid |I| = i}} \prod_{j \in I} x_j \right)$$

is the polar form of $F$.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$a_i$</th>
<th>$F(x)$</th>
<th>$f(x_1, \ldots, x_n)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$a_0 := 1, a_1 := 0$</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>$a_0 := 0, a_1 := 1$</td>
<td>$x$</td>
<td>$x_1$</td>
</tr>
<tr>
<td>2</td>
<td>$a_0 := 1, a_1 := 0, a_2 := 0$</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>$a_0 := 0, a_1 := 1, a_2 := 0$</td>
<td>$x$</td>
<td>$\frac{1}{2}(x_1 + x_2)$</td>
</tr>
<tr>
<td></td>
<td>$a_0 := 0, a_1 := 0, a_2 := 1$</td>
<td>$x^2$</td>
<td>$x_1 x_2$</td>
</tr>
<tr>
<td>3</td>
<td>$a_0 := 1, a_1 := 0, a_2 := 0, a_3 := 0$</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>$a_0 := 0, a_1 := 1, a_2 := 0, a_3 := 0$</td>
<td>$x$</td>
<td>$\frac{1}{3}(x_1 + x_2 + x_3)$</td>
</tr>
<tr>
<td></td>
<td>$a_0 := 0, a_1 := 0, a_2 := 1, a_3 := 0$</td>
<td>$x^2$</td>
<td>$\frac{1}{3}(x_1 x_2 + x_1 x_3 + x_2 x_3)$</td>
</tr>
<tr>
<td></td>
<td>$a_0 := 0, a_1 := 0, a_2 := 0, a_3 := 1$</td>
<td>$x^3$</td>
<td>$x_1 x_2 x_3$</td>
</tr>
</tbody>
</table>
Bernstein Polynomials and Polar Forms

Let \( F(x) := \begin{pmatrix} x \\ \frac{1}{2} x^2 \end{pmatrix} \). Hence \( f(x_1, x_2) = \begin{pmatrix} \frac{1}{2}(x_1 + x_2) \\ \frac{1}{2} x_1 x_2 \end{pmatrix} \), and we get

\[
\begin{align*}
f(0, 0) &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\
f(0, 1) &= \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix} = f(1, 0) \\
f(1, 1) &= \begin{pmatrix} 1 \\ \frac{1}{2} \end{pmatrix}.
\end{align*}
\]

Furthermore, \( F(t) = f(t, t) \), with

\[
\begin{align*}
f(t, t) &= f((1 - t) \cdot 0 + t \cdot 1, t) = (1 - t) \cdot f(0, t) + t \cdot f(1, t) \\
&= (1 - t)[(1 - t) \cdot f(0, 0) + t \cdot f(0, 1))] + t[(1 - t) \cdot f(1, 0) + t \cdot f(1, 1))] \\
&= (1 - t)^2 \cdot f(0, 0) + 2t(1 - t) \cdot f(0, 1) + t^2 \cdot f(1, 1) \\
&= B_{0,2}(t)f(0, 0) + B_{1,2}(t)f(0, 1) + B_{2,2}(t)f(1, 1).
\end{align*}
\]

Hence, there is a close connection between the polar form and the Bernstein polynomials: \( f(0, 0), f(0, 1), f(1, 1) \) form the coefficients (i.e., control points) of \( F \) relative to the Bernstein basis.
Bernstein Polynomials and Polar Forms

Lemma 97

Every polynomial can be expressed in Bezier form. That is, for every polynomial $P : \mathbb{R} \rightarrow \mathbb{R}^2$ of degree $n$ there exist control points $p_0, p_1, \ldots, p_n \in \mathbb{R}^2$ such that the Bezier curve defined by them matches $P|_{[0,1]}$.

Sketch of Proof: Let $f$ be the polarform of $P$, and let

$$p_k := f(0, \ldots, 0, 1, \ldots, 1)$$

for $k = 0, 1, \ldots, n$.

- Polar forms are useful because they provide a uniform and simple means for computing values of a polynomial using a variety of representations (Bezier, B-spline, NURBS, etc.).
- For this reason, some authors prefer to introduce Bezier curves in their polar form.
Derivatives of a Bézier Curve

Lemma 98

Let $B$ be a Bézier curve of degree $n$ with $n + 1$ control points $p_0, p_1, \ldots, p_n$. Its first derivative, which is sometimes called \textit{hodograph}, is a Bézier curve of degree $n - 1$,

$$B'(t) = \sum_{i=0}^{n-1} B_{i,n-1}(t) \left( n(p_{i+1} - p_i) \right),$$

whose $n$ control points are given by $n(p_1 - p_0), n(p_2 - p_1), \ldots, n(p_n - p_{n-1})$.

Proof: Since the control points are constants, computing the derivative of a Bézier curve is reduced to computing the derivatives of the Bernstein basis polynomials.

$$B'(t) = \frac{d}{dt} \left( \sum_{i=0}^{n} B_{i,n}(t)p_i \right) = \sum_{i=0}^{n} B'_{i,n}(t)p_i = n \left( \sum_{i=0}^{n} (B_{i-1,n-1}(t) - B_{i,n-1}(t))p_i \right)$$

$$= n \cdot \left( \sum_{i=1}^{n} B_{i-1,n-1}(t)p_i - \sum_{i=0}^{n-1} B_{i,n-1}(t)p_i \right)$$

$$= n \cdot \left( \sum_{i=0}^{n-1} B_{i,n-1}(t)p_{i+1} - \sum_{i=0}^{n-1} B_{i,n-1}(t)p_i \right) = \sum_{i=0}^{n-1} B_{i,n-1}(t) \left( n(p_{i+1} - p_i) \right) \quad \Box$$
Derivatives of a Bézier Curve

Lemma 99

A Bézier curve is tangent to the control polygon at the endpoints.

Proof: This is readily proved by computing $B'(0)$ and $B'(1)$. □

Hence, joining two Bézier curves in a $G^1$-continuous way is easy.

Let $p_0, p_1, \ldots, p_n$ and $p_0^*, p_1^*, \ldots, p_m^*$ be the control points of two Bézier curves $B$ and $B^*$. In order to achieve $C^1$-continuity, we need (in addition to $p_n = p_0^*$)

$$B'(1) = (B^*)'(0) \quad \text{i.e.,} \quad n(p_n - p_{n-1}) = m(p_1^* - p_0^*).$$

This has an interesting consequence for closed Bézier curves with $p_0 = B(0) = B(1) = p_n$:

- We get $G^1$-continuity at $p_0$ if $p_0, p_1, p_{n-1}$ are collinear.
- We get $C^1$-continuity at $p_0$ if $p_1 - p_0 = p_n - p_{n-1}$. 
Subdivision of a Bézier Curve

One can subdivide an $n$-degree Bézier curve $B$ into two curves, at a point $B(t_0)$ for a given parameter $t_0$, such that the newly obtained Bézier curves $B_1$ and $B_2$ have their own set of control points and are of degree $n$ each:

- First, we use de Casteljau’s algorithm to compute $B(t_0)$.
- The de Casteljau net can then be used to generate the new control polygons for $B_1$ and $B_2$.
- Note that $B_1$ and $B_2$ join in a $G^1$-continuous way.
Subdivision of a Bézier Curve

Lemma 100

Let \( p_0, p_1, \ldots, p_n \) be the control points of the Bézier curve \( \mathcal{B} \), and let \( p_{i,j} \) denote the control points obtained by de Casteljau’s algorithm for some \( t_0 \in ]0, 1[ \). We define new control points as follows:

\[
\begin{align*}
    p_i^* & := p_{i,0} \quad \text{for } i = 0, 1, \ldots, n \\
    p_i^{**} & := p_{n-i,i} \quad \text{for } i = 0, 1, \ldots, n
\end{align*}
\]

Let \( \mathcal{B}^* \) (\( \mathcal{B}^{**} \), resp.) denote the Bézier curve defined by \( p_0^*, p_1^*, \ldots, p_n^* \) (\( p_0^{**}, p_1^{**}, \ldots, p_n^{**} \), resp.). Then \( \mathcal{B}^* \) and \( \mathcal{B}^{**} \) join in a tangent-continuous way at point \( p_n^* = p_0^{**} \), and we have

\[
\mathcal{B}^* = \mathcal{B}|_{[0,t_0]} \quad \text{and} \quad \mathcal{B}^{**} = \mathcal{B}|_{[t_0,1]}.
\]

Note: With every subdivision the control polygons get closer to the Bézier curve. And the approximation is quite fast: For \( k \) subdivision steps, the maximum distance \( \varepsilon \) between the resulting control polygon and the curve is

\[
\varepsilon < \frac{c}{2^k}
\]

for some positive constant \( c \).
Degree Elevation of a Bézier Curve

- An increase of the number of control points of a Bézier curve increases the flexibility in designing shapes.
- The key goal is to preserve the shape of the curve. (Recall that Bézier curves change globally if one control point is relocated!)
- Of course, adding one control point means increasing the degree of a Bézier curve by one.
- Let $p_0, p_1, \ldots, p_n$ be the old control points, and $p^*_0, p^*_1, \ldots, p^*_n, p^*_{n+1}$ be the new control points, and denote the Bézier curves defined by them by $B$ and $B^*$.
- How can we guarantee $B(t) = B^*(t)$ for all $t \in [0, 1]$?
- Obviously, we will need $p_0 = p^*_0$ and $p_n = p^*_n$ in order to ensure that at least the start and end points of $B$ and $B^*$ match.
- In the sequel, we will find it convenient to extend the index range of the control points of $B$ and introduce (arbitrary) points $p_{-1}$ and $p_{n+1}$. (Both points will be multiplied with factors that equal zero, anyway.)
Degree Elevation of a Bézier Curve

- Standard equalities:

\[
\binom{n+1}{i}(1-t) \cdot B_{i,n}(t) = \binom{n+1}{i}(1-t) \binom{n}{i} t^i(1-t)^{n-i}
\]

\[
= \binom{n}{i} \binom{n+1}{i} t^i(1-t)^{n+1-i} = \binom{n}{i} B_{i,n+1}(t)
\]

and

\[
\binom{n+1}{i+1} t \cdot B_{i,n}(t) = \binom{n+1}{i+1} t \binom{n}{i} t^i(1-t)^{n-i}
\]

\[
= \binom{n}{i} \binom{n+1}{i+1} t^{i+1}(1-t)^{n-i} = \binom{n}{i} B_{i+1,n+1}(t)
\]

- Hence,

\[
(1-t) \cdot B_{i,n}(t) = \frac{n+1-i}{n+1} B_{i,n+1}(t) \quad \text{and} \quad t \cdot B_{i,n}(t) = \frac{i+1}{n+1} B_{i+1,n+1}(t).
\]
Degree Elevation of a Bézier Curve

\[ B(t) = \sum_{i=0}^{n} B_{i,n}(t)p_i = ((1 - t) + t) \sum_{i=0}^{n} B_{i,n}(t)p_i \]

\[ = (1 - t) \sum_{i=0}^{n} B_{i,n}(t)p_i + t \sum_{i=0}^{n} B_{i,n}(t)p_i = \sum_{i=0}^{n} (1 - t) \cdot B_{i,n}(t)p_i + \sum_{i=0}^{n} t \cdot B_{i,n}(t)p_i \]

\[ = \sum_{i=0}^{n} \frac{n + 1 - i}{n + 1} B_{i,n+1}(t)p_i + \sum_{i=0}^{n} \frac{i + 1}{n + 1} B_{i+1,n+1}(t)p_i \]

\[ = \sum_{i=0}^{n+1} \frac{n + 1 - i}{n + 1} B_{i,n+1}(t)p_i + \sum_{i=0}^{n+1} \frac{i + 1}{n + 1} B_{i+1,n+1}(t)p_i \]

\[ = \sum_{i=0}^{n+1} \frac{n + 1 - i}{n + 1} B_{i,n+1}(t)p_i + \sum_{i=0}^{n+1} \frac{i}{n + 1} B_{i,n+1}(t)p_{i-1} \]

\[ = \sum_{i=0}^{n+1} B_{i,n+1}(t) \left( \frac{i}{n + 1} p_{i-1} + \frac{n + 1 - i}{n + 1} p_i \right) = \sum_{i=0}^{n+1} B_{i,n+1}(t)p_i^* =: B^*(t) \]

with

\[ p_i^* := \frac{i}{n + 1} p_{i-1} + \frac{n + 1 - i}{n + 1} p_i, \quad i = 0, \ldots, n + 1. \]
Degree Elevation of a Bézier Curve

Lemma 101

Let $p_0, p_1, \ldots, p_n$ be the control points of the degree-$n$ Bézier curve $B$. If we use

$$p_i^* := \left( \frac{i}{n+1} \right) p_{i-1} + \left( 1 - \frac{i}{n+1} \right) p_i \quad \text{for } i = 0, 1, \ldots, n+1$$

as control points for the Bézier curve $B^*$ of degree $n + 1$, then

$$B(t) = B^*(t) \quad \text{for all } t \in [0, 1].$$
Degree Elevation of a Bézier Curve

- Note that all newly created control points lie on the edges of the previous control polygon.
- Effectively, the corners of the previous control polygon are cut off.
- Degree elevation can be used repeatedly, e.g., in order to arrive at the same degrees for two Bézier curves that join.
- As the degree keeps increasing, the control polygon approaches the Bézier curve and has it as a limiting position.
Bézier Surfaces

Definition 102 (Bézier surface)

Suppose that we are given a set of \((n + 1) \cdot (m + 1)\) control points in \(\mathbb{R}^3\), with \(0 \leq i \leq n\) and \(0 \leq j \leq m\), where the control point on the \(i\)-th row and \(j\)-th column is denoted by \(p_{i,j}\). The Bézier surface \(S : [0, 1] \times [0, 1] \rightarrow \mathbb{R}^3\) defined by \(p_{i,j}\) is given by

\[
S(u, v) := \sum_{i=0}^{n} \sum_{j=0}^{m} B_{i,n}(u) B_{j,m}(v) p_{i,j} \quad \text{for } (u, v) \in [0, 1] \times [0, 1],
\]

where \(B_{k,d}(x) := \binom{d}{k} x^k (1 - x)^{d-k}\) is the \(k\)-th Bernstein basis polynomial of degree \(d\).

- Since \(B_{i,n}(u)\) and \(B_{j,m}(v)\) are polynomials of degree \(n\) and \(m\), this is called a Bézier surface of degree \((n, m)\).
- The set of control points is called a Bézier net or control net.
Properties of Bézier Surfaces

Lemma 103

For all \( n, m \in \mathbb{N}_0 \) and all \( 0 \leq i \leq n \) and \( 0 \leq j \leq m \), and all \( (u, v) \in [0, 1] \times [0, 1] \), the term \( B_{i,n}(u)B_{j,m}(v) \) is non-negative.

Lemma 104 (Partition of unity)

For all \( m, n \in \mathbb{N}_0 \), the sum of all \( B_{i,n}(u)B_{j,m}(v) \) is one:

\[
\sum_{i=0}^{n} \sum_{j=0}^{m} B_{i,n}(u)B_{j,m}(v) = 1 \quad \text{for all } (u, v) \in [0, 1] \times [0, 1].
\]

Proof: We have for all \( m, n \in \mathbb{N}_0 \) and all \( (u, v) \in [0, 1] \times [0, 1] \)

\[
\sum_{i=0}^{n} \sum_{j=0}^{m} B_{i,n}(u)B_{j,m}(v) = \sum_{i=0}^{n} B_{i,n}(u) \left( \sum_{j=0}^{m} B_{j,m}(v) \right) = \sum_{i=0}^{n} B_{i,n}(u) = 1.
\]
Properties of Bézier Surfaces

Lemma 105 *(Convex hull property)*

A Bézier surface lies completely inside the convex hull of its control points.

**Proof:** Recall that $S(u, v)$ is the linear combination of all its control points with non-negative coefficients whose sum is one. \qed

Lemma 106

A Bézier surface passes through the four corners $p_{0,0}, p_{n,0}, p_{0,m}$ and $p_{n,m}$.

**Proof:** Recall that

$$B_{i,n}(0) = \begin{cases} 1 & \text{for } i = 0, \\ 0 & \text{for } i > 0, \end{cases} \quad \text{and} \quad B_{j,m}(0) = \begin{cases} 1 & \text{for } j = 0, \\ 0 & \text{for } j > 0. \end{cases}$$

Hence, $S(0, 0) = B_{0,n}(0)B_{0,m}(0)p_{0,0} = p_{0,0}$. Similarly for the other corners. \qed

Lemma 107

Applying an affine transformation to the control points results in the same transformation as obtained by transforming the surface’s equation.
Lemma 108

Consider a Bézier surface $S: [0, 1] \times [0, 1] \rightarrow \mathbb{R}^3$ defined by $(n + 1) \cdot (m + 1)$ control points $p_{i,j}$, with $0 \leq i \leq n$ and $0 \leq j \leq m$, and let $v_0 \in [0, 1]$ be fixed. Then $C: [0, 1] \rightarrow \mathbb{R}^3$ defined as

$$C(u) := \sum_{i=0}^{n} \sum_{j=0}^{m} B_{i,n}(u) B_{j,m}(v_0) p_{i,j} \quad \text{for } u \in [0, 1]$$

is a Bézier curve defined by the $n + 1$ control points $q_0, q_1, \ldots, q_n \in \mathbb{R}^3$, where

$$q_i := \sum_{j=0}^{m} B_{j,m}(v_0) p_{i,j} \quad \text{for } 0 \leq i \leq n.$$

Proof: We have for all $u \in [0, 1]$

$$C(u) = \sum_{i=0}^{n} \sum_{j=0}^{m} B_{i,n}(u) B_{j,m}(v_0) p_{i,j} = \sum_{i=0}^{n} B_{i,n}(u) \left( \sum_{j=0}^{m} B_{j,m}(v_0) p_{i,j} \right) = \sum_{i=0}^{n} B_{i,n}(u) q_i.$$

▶ Analogously for fixed $u_0$. 
Isoparametric Curves of Bézier Surfaces

Corollary 109

The boundary curves of a Bézier surface are Bézier curves defined by the boundary points of its control net.

Lemma 110 (*Tangency in the corner points*)

Consider a Bézier surface $S: [0, 1] \times [0, 1] \rightarrow \mathbb{R}^3$ defined by $(n + 1) \cdot (m + 1)$ control points $p_{i,j}$, with $0 \leq i \leq n$ and $0 \leq j \leq m$. The tangent plane at $S(0, 0) = p_{0,0}$ is spanned by the vectors $p_{1,0} - p_{0,0}$ and $p_{0,1} - p_{0,0}$. 
Bézier Surface as Tensor-Product Surface

- A Bézier surface is generated by “multiplying” two Bézier curves: tensor product surface.

Lemma 111

Consider a Bézier surface \( S : [0, 1] \times [0, 1] \rightarrow \mathbb{R}^3 \) defined by \((n + 1) \cdot (m + 1)\) control points \( p_{i,j} \), with \( 0 \leq i \leq n \) and \( 0 \leq j \leq m \). Then \( S \) is a tensor-product surface:

\[
S(u, v) = (B_{0,n}(u), B_{1,n}(u), \ldots, B_{n,n}(u)) \cdot \\
\begin{pmatrix}
p_{0,0} & p_{0,1} & \cdots & p_{0,m} \\
p_{1,0} & p_{1,1} & \cdots & p_{1,m} \\
\vdots & \vdots & \ddots & \vdots \\
p_{n,0} & p_{n,1} & \cdots & p_{n,m}
\end{pmatrix} \cdot \\
\begin{pmatrix}
B_{0,m}(v) \\
B_{1,m}(v) \\
\vdots \\
B_{m,m}(v)
\end{pmatrix}
\]

**Proof:** Just do the math!
B-Spline Curves and Surfaces

- Shortcomings of Bézier Curves
- B-Spline Basis Functions
- B-Spline Curves
- B-Spline Surfaces
- Non-Uniform Rational B-Spline Curves and Surfaces
Shortcomings of Bézier Curves

- Modifying the vertex $p_j$ of a Bézier curve causes a global change of the entire curve:

  \[ B^*(t) = B(t) + B_{j,n}(t)v \]

- But $B_{j,n}(t) \neq 0$ for all $t$ with $0 < t < 1$!
Shortcomings of Bézier Curves

- While it is easy to join two Bézier curves with $G^1$ continuity, achieving $C^2$ or even higher continuity is quite cumbersome.

$$p_2, p_3 = p_0^*, p_1^* \text{ collinear}$$
Shortcomings of Bézier Curves

- While it is easy to join two Bézier curves with $G^1$ continuity, achieving $C^2$ or even higher continuity is quite cumbersome.
- Even worse, changing the common end point of two consecutive Bézier curves destroys $G^1$ continuity.
Shortcomings of Bézier Curves

- While it is easy to join two Bézier curves with $G^1$ continuity, achieving $C^2$ or even higher continuity is quite cumbersome.
- This will be easier for B-spline curves. (Depicted are two cubic B-splines.)
Shortcomings of Bézier Curves

- While it is easy to join two Bézier curves with $G^1$ continuity, achieving $C^2$ or even higher continuity is quite cumbersome.
- This will be easier for B-spline curves. (Depicted are two cubic B-splines.)
Shortcomings of Bézier Curves

- It is fairly difficult to squeeze a Bézier curve close to a sharp corner of the control polygon.
Shortcomings of Bézier Curves

- Adding additional control vertices hardly helps but increases the degree of the Bézier curve, which may result in oscillation and cause numerical instability.

three new vertices
Introduction to B-Splines

- Curves consisting of just one segment have several drawbacks:
  - To satisfy all given constraints, often a high polynomial degree is required.
  - The number of control points is directly related to the degree.
  - Interactive shape design is inaccurate or requires high computational costs.
- The solution is to use a sequence of polynomial or rational curves to form one continuous curve: spline.
- Historically, the term spline (Dt.: Straklatte) was used for elastic wooden strips in the shipbuilding industry, which pass through given constrained points called ducks such that the strain of the strip is minimized.
  
  ![Diagram of ducks and spline]

- Mathematical splines were introduced by Isaac Jacob Schoenberg in 1946.

Warning
The terminology and the definitions used for B-splines vary from author to author! Thus, make sure to check carefully the definitions given in textbooks or research papers.
Definition 112 (*Spline*)

A curve \( C(t) : [a, b] \rightarrow \mathbb{R}^2 \) is called a *spline* of degree \( k \) (and order \( k + 1 \)), for \( k \in \mathbb{N} \), if there exist

- \( m \) polynomials \( P_1, P_2, \ldots, P_m \) of degree \( k \), for some \( m \in \mathbb{N} \), and
- \( m + 1 \) parameters \( t_0, \ldots, t_m \in \mathbb{R} \)

such that

1. \( a = t_0 < t_1 < \ldots < t_{m-1} < t_m = b \),
2. \( C|_{[t_{i-1}, t_i]} = P_i|_{[t_{i-1}, t_i]} \) for all \( i \in \{1, 2, \ldots, m\} \).
Introduction to B-Splines

▶ The numbers $t_0, \ldots, t_m$ are called breakpoints or knots.
▶ The definition implies

$$P_i(t_i) = P_{i+1}(t_i) \quad \text{for all } i \in \{1, 2, \ldots, m - 1\}.$$ 

▶ Special case $k = 1$: We get a polygonal curve.

▶ The polynomials join with some unknown degree of continuity at the breakpoints. (We have at least $C^0$-continuity.)
▶ Obvious problem: How can we achieve a reasonable degree of continuity?
Knot Vector

Definition 113 (*Knot vector*, *Dt.*: *Knotenvektor*)

In general, a knot vector is a sequence of non-decreasing real numbers. A **finite knot vector** is a sequence of \( m + 1 \) real numbers \( \tau := (t_0, t_1, t_2, \ldots, t_m) \), for some \( m \in \mathbb{N} \), such that \( t_i \leq t_{i+1} \) for all \( 0 \leq i < m \).

An **infinite knot vector** is a sequence of real numbers \( \tau := (t_0, t_1, t_2, \ldots) \) such that \( t_i \leq t_{i+1} \) for all \( i \in \mathbb{N}_0 \).

A **bi-infinite knot vector** is a sequence of real numbers \( \tau := (\ldots, t_{-2}, t_{-1}, t_0, t_1, t_2, \ldots) \) such that \( t_i \leq t_{i+1} \) for all \( i \in \mathbb{Z} \).

The numbers \( t_i \) are called **knots**, and the **\( i \)-th knot span** is given by the (half-open) interval \([t_i, t_{i+1}) \subset \mathbb{R}\).

For (bi)infinite knot vectors we assume \( \sup_{i \to \infty} t_i = \infty \) and \( \inf_{i \to -\infty} t_i = -\infty \).

For some of the subsequent definitions we will find it convenient to deal with (bi)infinite knot vectors. With some extra care for “boundary conditions” one could replace all (bi)infinite knot vectors by finite knot vectors.
Knot Vector

Definition 114 (Multiplicity of a knot, Dt.: Vielfachheit eines Knotens)

Let $\tau$ be a finite or (bi)infinite knot vector. If a knot $t_i$ appears exactly $k > 1$ times in $\tau$, for a permissible value of $i \in \mathbb{Z}$, i.e., if $t_{i-1} < t_i = t_{i+1} = \cdots = t_{i+k-1} < t_{i+k}$, then $t_i$ is a multiple knot of multiplicity $k$. Otherwise, if $t_i$ appears only once in $\tau$ then $t_i$ is a simple knot.

Definition 115 (Uniform knot vector)

A finite or (bi)infinite knot vector is uniform if there exists $c \in \mathbb{R}^+$ such that $t_{i+1} - t_i = c$ for all (permissible) values of $i \in \mathbb{Z}$, except for possibly the first and last knots of higher multiplicity in case of a finite knot sequence. Otherwise, the knot vector is non-uniform.
B-Spline Basis Functions

- We define the B-spline basis functions analytically, using the recurrence formula by de Boor, Cox and Mansfield.

**Definition 116 (B-spline basis function)**

Let $\tau$ be a finite or (bi)infinite knot vector. For all (permissible) $i \in \mathbb{Z}$ and $k \in \mathbb{N}_0$, the $i$-th B-spline basis function, $N_{i,k,\tau}(t)$, of degree $k$ (and order $k + 1$) relative to $\tau$ is defined as,

if $k = 0$,

$$N_{i,0,\tau}(t) = \begin{cases} 
1 & \text{if } t_i \leq t < t_{i+1}, \\
0 & \text{otherwise,}
\end{cases}$$

and if $k > 0$ as

$$N_{i,k,\tau}(t) = \frac{t - t_i}{t_{i+k} - t_i} N_{i,k-1,\tau}(t) + \frac{t_{i+k+1} - t}{t_{i+k+1} - t_{i+1}} N_{i+1,k-1,\tau}(t).$$

- In case of multiple knots, indeterminate terms of the form $0/0$ are taken as zero!
- Alternatively, one can demand $t_i < t_{i+k}$ for all (permissible) $i \in \mathbb{Z}$.
- Aka: Normalized B(asic)-Spline Blending Functions.
Plugging into the definition yields

\[ N_{i,1,\tau}(t) = \frac{t - t_i}{t_{i+1} - t_i} N_{i,0,\tau}(t) + \frac{t_{i+2} - t}{t_{i+2} - t_{i+1}} N_{i+1,0,\tau}(t) \]

\[
= \begin{cases} 
0 & \text{if } t \notin [t_i, t_{i+2}], \\
\frac{t - t_i}{t_{i+1} - t_i} & \text{if } t \in [t_i, t_{i+1}], \\
\frac{t_{i+2} - t}{t_{i+2} - t_{i+1}} & \text{if } t \in [t_{i+1}, t_{i+2}]. 
\end{cases}
\]

The functions \( N_{i,1,\tau}(t) \) are called hat functions or chapeau functions. They are widely used in signal processing and finite-element techniques.

Note that \( N_{i,1,\tau}(t) \) is continuous at \( t_{i+1} \).

For a uniform knot vector \( \tau \) with \( c := t_{i+1} - t_i \) this simplifies to

\[
N_{i,1,\tau}(t) = \begin{cases} 
0 & \text{if } t \notin [t_i, t_{i+2}], \\
\frac{1}{c}(t - t_i) & \text{if } t \in [t_i, t_{i+1}], \\
\frac{1}{c}(t_{i+2} - t) & \text{if } t \in [t_{i+1}, t_{i+2}]. 
\end{cases}
\]
B-Spline Basis Functions

- Basis functions $N_{i,k,\tau}$.

$N_{1,0}$ is a step function that is 1 over the knot span $[t_i, t_{i+1}]$.

$N_{1,1}$ is a piecewise linear function that is non-zero over two knot spans $[t_i, t_{i+2}]$ and goes from 0 to 1 and back.

$N_{1,2}$ is a piecewise quadratic function that is non-zero over three knot spans $[t_i, t_{i+3}]$. 

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Sample B-Spline Basis Functions

- Basis functions of degree 0:

\[ N_{0,0}, \quad N_{1,0}, \quad N_{2,0}, \quad N_{3,0}, \quad N_{4,0} \]
Sample B-Spline Basis Functions

- Basis functions of degree 1:
Sample B-Spline Basis Functions

- Basis functions of degree 2:

\[ N_{0,2} \]

\[ N_{2,2} \]

\[ N_{1,2} \]
Sample B-Spline Basis Functions

- Uniform knot vector \( (0, \frac{1}{9}, \frac{2}{9}, \frac{3}{9}, \frac{4}{9}, \frac{5}{9}, \frac{6}{9}, \frac{7}{9}, \frac{8}{9}, 1) \) with ten knots.

\[
\begin{align*}
N_{0,0}, N_{1,0}, \ldots, N_{8,0} \\
N_{0,1}, N_{1,1}, \ldots, N_{7,1} \\
N_{0,2}, N_{1,2}, \ldots, N_{6,2} \\
N_{0,3}, N_{1,3}, \ldots, N_{5,3}
\end{align*}
\]
Sample B-Spline Basis Functions

- Clamped uniform knot vector \( (0, 0, 0, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, 1, 1, 1) \) with ten knots.
Sample B-Spline Basis Functions

- Non-uniform knot vector \( (0, \frac{1}{6}, \frac{1}{3}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{2}{3}, \frac{5}{6}, 1) \) with ten knots.

\[ N_{0,0}, N_{1,0}, N_{2,0}, N_{6,0}, N_{7,0}, N_{8,0} \]

\[ N_{0,1}, N_{1,1}, N_{2,1}, N_{5,1}, N_{6,1}, N_{7,1} \]

\[ N_{0,2}, N_{1,2}, N_{2,2}, N_{4,2}, N_{5,2}, N_{6,2} \]

\[ N_{0,3}, N_{1,3}, \ldots, N_{5,3} \]
Properties of B-Spline Basis Functions

- It is common to omit the explicit mentioning of the dependency of \( N_{i,k,\tau}(t) \) on \( \tau \), and to write \( N_{i,k}(t) \). (And sometimes we simply write \( N_{i,k} \) . . .)

- For \( k > 0 \), each \( N_{i,k,\tau}(t) \) is a linear combination of two B-spline basis functions of degree \( k - 1 \): \( N_{i,k-1,\tau}(t) \) and \( N_{i+1,k-1,\tau}(t) \).

- This suggests a recursive analysis of the dependencies.

\[
\begin{array}{cccccc}
N_{0,0} & N_{0,1} & N_{0,2} & N_{0,3} & N_{0,4} \\
N_{1,0} & N_{1,1} & N_{1,2} & N_{1,3} \\
N_{2,0} & N_{2,1} & N_{2,2} \\
N_{3,0} & N_{3,1} \\
N_{4,0} & \\
\end{array}
\]
Properties of B-Spline Basis Functions

- It is common to omit the explicit mentioning of the dependency of $N_{i,k,\tau}(t)$ on $\tau$, and to write $N_{i,k}(t)$. (And sometimes we simply write $N_{i,k}$ ...)

- For $k > 0$, each $N_{i,k,\tau}(t)$ is a linear combination of two B-spline basis functions of degree $k - 1$: $N_{i,k-1,\tau}(t)$ and $N_{i+1,k-1,\tau}(t)$.

- $N_{i,k,\tau}(t)$ depends on $N_{i,0,\tau}(t)$, $N_{i+1,0,\tau}(t)$, ..., $N_{i+k,0,\tau}(t)$. 

\[
egin{array}{c}
N_{0,0} & N_{0,1} & N_{0,2} & N_{0,3} & N_{0,4} \\
N_{1,0} & N_{1,1} & N_{1,2} & N_{1,3} & \\
N_{2,0} & N_{2,1} & N_{2,2} & & \\
N_{3,0} & N_{3,1} & & & \\
N_{4,0} & & & & \\
\end{array}
\]
Properties of B-Spline Basis Functions

- It is common to omit the explicit mentioning of the dependency of $N_{i,k,\tau}(t)$ on $\tau$, and to write $N_{i,k}(t)$. (And sometimes we simply write $N_{i,k}$ ...)
- For $k > 0$, each $N_{i,k,\tau}(t)$ is a linear combination of two B-spline basis functions of degree $k - 1$: $N_{i,k-1,\tau}(t)$ and $N_{i+1,k-1,\tau}(t)$.
- $N_{i,k,\tau}(t)$ is non-zero only for $t \in [t_i, t_{i+k+1}[$.
Lemma 117 (*Local support, Dt.: lokaler Träger*)

Let \( \tau := (\ldots, t_{-2}, t_{-1}, t_0, t_1, t_2, \ldots) \) be a (bi)infinite knot vector. For all (permissible) \( i \in \mathbb{Z} \) and \( k \in \mathbb{N}_0 \) we have

\[
N_{i,k,\tau}(t) = 0 \quad \text{if } t \notin [t_i, t_{i+k+1}].
\]

**Proof:** We do a proof by induction on \( k \).

I.B.: By definition, this claim is correct for \( k = 0 \) and all (permissible) \( i \in \mathbb{Z} \).

I.H.: Suppose that it is true for all basis functions of degree \( k - 1 \), for some arbitrary but fixed \( k \in \mathbb{N} \). I.e., \( N_{i,k-1,\tau}(t) = 0 \) if \( t \notin [t_i, t_{i+k}], \) for all (permissible) \( i \in \mathbb{Z} \).

I.S.: Recall that

\[
N_{i,k,\tau}(t) = \frac{t - t_i}{t_{i+k} - t_i} N_{i,k-1,\tau}(t) + \frac{t_{i+k+1} - t}{t_{i+k+1} - t_{i+1}} N_{i+1,k-1,\tau}(t).
\]

Hence, \( N_{i,k,\tau}(t) = 0 \) if \( t \notin ([t_i, t_{i+k}] \cup [t_{i+1}, t_{i+k+1}]), \) i.e., if \( t \notin [t_i, t_{i+k+1}] \). \( \square \)
Properties of B-Spline Basis Functions

Lemma 118 (*Non-negativity*)

We have $N_{i,k,\tau}(t) \geq 0$ for all (permissible) $i \in \mathbb{Z}$ and $k \in \mathbb{N}_0$, and all real $t$.

*Proof*: Again we do a proof by induction on $k$.

I.B.: By definition, this claim is correct for $k = 0$ and all (permissible) $i \in \mathbb{Z}$.

I.H.: Suppose that it is true for all basis functions of degree $k - 1$, for some arbitrary but fixed $k \in \mathbb{N}$.

I.S.: Lemma 117 tells us that $N_{i,k,\tau}(t) = 0$ if $t \notin [t_i, t_{i+k+1}]$. Hence, we can focus on $t \in [t_i, t_{i+k+1}]$ and get

$$N_{i,k,\tau}(t) = \frac{t - t_i}{t_{i+k} - t_i} \cdot N_{i,k-1,\tau}(t) + \frac{t_{i+k+1} - t}{t_{i+k+1} - t_i} \cdot N_{i+1,k-1,\tau}(t) \geq 0 \text{ (I.H.)}$$

$\geq 0$.

Lemma 119

For all $k \in \mathbb{N}$, all B-spline basis functions of degree $k$ are continuous.
Properties of B-Spline Basis Functions

Lemma 120

Let $\tau := (\ldots, t_{-2}, t_{-1}, t_0, t_1, t_2, \ldots)$ be a (bi)infinite knot vector. For all (permissible) $i \in \mathbb{Z}$ and $k \in \mathbb{N}_0$, the basis functions

$$N_{i-k,k,\tau}(t), \ N_{i-k+1,k,\tau}(t), \ldots, \ N_{i,k,\tau}(t)$$

are the only (at most) $k + 1$ basis functions of degree $k$ that are (possibly) non-zero over the interval $[t_i, t_{i+1}]$. 

\[\begin{array}{c c c}
[t_0, t_1] & N_{0,0} & \ & \ & \ & N_{0,0} \\
[t_1, t_2] & N_{1,0} & N_{0,1} & N_{0,2} & N_{0,3} & N_{0,4} \\
[t_2, t_3] & N_{2,0} & N_{1,1} & N_{1,2} & N_{1,3} \\
[t_3, t_4] & N_{3,0} & N_{2,1} & N_{2,2} \\
[t_4, t_5] & N_{4,0} & N_{3,1} \\
\end{array}\]
Properties of B-Spline Basis Functions

Lemma 120

Let $\tau := (\ldots, t_{-2}, t_{-1}, t_0, t_1, t_2, \ldots)$ be a (bi)infinite knot vector. For all (permissible) $i \in \mathbb{Z}$ and $k \in \mathbb{N}_0$, the basis functions

$$N_{i-k,k,\tau}(t), N_{i-k+1,k,\tau}(t), \ldots, N_{i,k,\tau}(t)$$

are the only (at most) $k + 1$ basis functions of degree $k$ that are (possibly) non-zero over the interval $[t_i, t_{i+1}]$.

Proof: The Local Support Lemma 117 tells us that

$$N_{j,k,\tau}(t) = 0 \quad \text{if } t \notin [t_j, t_{j+k+1}]$$

and, thus, possibly non-zero only if $t \in [t_j, t_{j+k+1}]$. Hence, $N_{j,k,\tau}(t) \neq 0$ over $[t_i, t_{i+1}]$ only if $i \geq j$ and $i + 1 \leq j + k + 1$, i.e., if $j \leq i$ and $j \geq i - k$. Thus,

$$N_{i-k,k,\tau}(t), N_{i-k+1,k,\tau}(t), \ldots, N_{i,k,\tau}(t)$$

are the only B-spline basis functions that are (possibly) non-zero over $[t_i, t_{i+1}]$. 

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Properties of B-Spline Basis Functions

Lemma 121

All B-spline basis functions of degree $k$ are piecewise polynomials of degree $k$.

\[ N_{i,1}, \quad N_{1,2} \]

Lemma 122

All B-spline basis functions of degree $k$ are $k - r$ times continuously differentiable at a knot of multiplicity $r$, and $k - 1$ times continuously differentiable everywhere else. The first derivative of $N_{i,k}(t)$ is given as follows:

\[
N'_{i,k}(t) = \frac{k}{t_{i+k} - t_i} N_{i,k-1}(t) - \frac{k}{t_{i+k+1} - t_{i+1}} N_{i+1,k-1}(t)
\]
Properties of B-Spline Basis Functions

Lemma 123
For a uniform knot vector $\tau$ all B-spline basis functions of the same degree are shifted copies of each other: For all $t \in \mathbb{R}$ and all (permissible) $i \in \mathbb{Z}$ and $k \in \mathbb{N}_0$ we have $N_{i,k,\tau}(t) = N_{0,k,\tau}(t - i \cdot c)$, where $c := t_1 - t_0$.

Lemma 124 (Partition of unity, Dt.: Zerlegung der Eins)
Let $\tau = (t_0, t_1, t_2, \ldots, t_m)$ be a finite knot vector, and $k \in \mathbb{N}_0$ with $k < \frac{m}{2}$. Then,

$$\sum_{i=0}^{m-k-1} N_{i,k,\tau}(t) = 1 \quad \text{for all } t \in [t_k, t_{m-k}].$$

Corollary 125
Let $\tau = (t_0, t_1, t_2, \ldots, t_{n+k+1})$ be a finite knot vector, for some $k \in \mathbb{N}_0$ with $k \leq n$. Then,

$$\sum_{i=0}^{n} N_{i,k,\tau}(t) = 1 \quad \text{for all } t \in [t_k, t_{n+1}].$$
Properties of B-Spline Basis Functions

Proof of Lemma 124 (Partition of Unity): We do a proof by induction on \( k \).

I.B.: By definition, this claim is correct for \( k = 0 \).

I.H.: Suppose that it is true for degree \( k - 1 \), for some arbitrary but fixed \( k \in \mathbb{N} \) such that \( k < \frac{m}{2} \). I.e., suppose that \( \sum_{i=0}^{m-k} N_{i,k-1,\tau}(t) = 1 \) for all \( t \in [t_{k-1}, t_{m-k+1}] \).

I.S.: Recall that (by Lem. 117)
\[
N_{0,k-1,\tau}(t) = 0 \text{ for } t \notin [t_0, t_k] \quad \text{and} \quad N_{m-k,k-1,\tau}(t) = 0 \text{ for } t \notin [t_{m-k}, t_m].
\]

Let \( t \in [t_k, t_{m-k}] \) be arbitrary but fixed. Applying the recursion yields
\[
\sum_{i=0}^{m-k-1} N_{i,k,\tau}(t) = \sum_{i=0}^{m-k-1} \left( \frac{t - t_i}{t_{i+k} - t_i} N_{i,k-1,\tau}(t) + \frac{t_{i+k+1} - t}{t_{i+k+1} - t_{i+1}} N_{i+1,k-1,\tau}(t) \right)
= \sum_{i=1}^{m-k-1} \frac{t - t_i}{t_{i+k} - t_i} N_{i,k-1,\tau}(t) + \sum_{i=0}^{m-k-2} \frac{t_{i+k+1} - t}{t_{i+k+1} - t_{i+1}} N_{i+1,k-1,\tau}(t)
= \sum_{i=1}^{m-k-1} \frac{t - t_i}{t_{i+k} - t_i} N_{i,k-1,\tau}(t) + \sum_{i=1}^{m-k-1} \frac{t_{i+k} - t}{t_{i+k} - t_i} N_{i,k-1,\tau}(t)
= \sum_{i=1}^{m-k-1} N_{i,k-1,\tau}(t) = \sum_{i=0}^{m-k} N_{i,k-1,\tau}(t) \quad \forall t \in [t_{k-1}, t_{m-k+1}],
\]

\[= 1.\]
B-Spline Curves

Definition 126 (*B-spline curve*)

For $n \in \mathbb{N}$ and $k \in \mathbb{N}_0$ with $k \leq n$, consider a set of $n + 1$ control points with position vectors $p_0, p_1, \ldots, p_n$ in the plane, and let $\tau := (t_0, t_1, \ldots, t_{n+k+1})$ be a knot vector. Then the *B-spline curve* of degree $k$ (and order $k + 1$) relative to $\tau$ with control points $p_0, p_1, \ldots, p_n$ is given by

$$P(t) := \sum_{i=0}^{n} N_{i,k,\tau}(t)p_i \quad \text{for } t \in [t_k, t_{n+1}],$$

where $N_{i,k,\tau}$ is the $i$-th B-spline basis function of degree $k$ relative to $\tau$.

- The degree $k$ is (except for $k \leq n$) independent of the number $n + 1$ of control points!
- The restriction of $t$ to the interval $[t_k, t_{n+1}]$ guarantees that the basis functions sum up to 1 for all (permissible) values of $t$. (Recall the Partition of Unity, Cor. 125.)
Clamped and Unclamped B-Spline Curves

Definition 127 \textit{(Clamped B-spline)}

Let $\mathcal{P}$ be a B-spline curve of degree $k$ defined by $n + 1$ control points with position vectors $p_0, p_1, \ldots, p_n$, over the knot vector $\tau := (t_0, t_1, \ldots, t_{n+k+1})$, for $n \in \mathbb{N}$ and $k \in \mathbb{N}_0$ with $k \leq n$. If $t_0 = t_1 = \ldots = t_k < t_{k+1}$ and $t_n < t_{n+1} = \ldots = t_{n+k+1}$ then we say that the knot vector and the B-spline curve are \textit{clamped}.

- Recall that the Partition of Unity (Cor. 125) holds for all $t \in [t_k, t_{n+1}]$.
- Typically, for a clamped knot vector,

$$0 = t_0 = t_1 = \ldots = t_k \quad \text{and} \quad t_{n+1} = \ldots = t_{n+k+1} = 1.$$  

Lemma 128

Let $\mathcal{P}$ be a B-spline curve of degree $k$ defined by $n + 1$ control points with position vectors $p_0, p_1, \ldots, p_n$, over the clamped knot vector $\tau := (t_0, t_1, \ldots, t_{n+k+1})$, for $n \in \mathbb{N}$ and $k \in \mathbb{N}_0$ with $k \leq n$. Then $\mathcal{P}$ starts in $p_0$ and ends in $p_n$. 
Clamped and Unclamped B-Spline Curves

▶ Control points: \( \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 3 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \\ 6 \\ 8 \\ 8 \end{pmatrix} \right\} \).

uniform unclamped cubic B-spline: \( \tau = (0, \frac{1}{10}, \frac{1}{5}, \frac{3}{10}, \frac{3}{5}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{7}{10}, \frac{4}{5}, \frac{9}{10}, 1) \)

uniform clamped cubic B-spline: \( \tau = (0, 0, 0, 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1, 1, 1, 1) \).
Proof of Lemma 128

We prove $P(t_k) = p_0$. Recall that $N_{0,k}(t)$ is non-zero only for $t \in [t_0, t_{k+1}]$.

However, for a clamped knot vector with $t_0 = t_1 = \ldots = t_k < t_{k+1}$ we have

$$N_{0,0}(t) = N_{1,0}(t) = \ldots = N_{k-1,0}(t) = 0 \quad \text{for all } t, \quad \text{and } N_{k,0}(t_k) = 1.$$ 

The recursion formula for the B-spline basis functions yields

$$N_{i,j}(t) = 0 \quad \text{for all } i, j \text{ with } i + j \leq k - 1 \text{ and for all } t.$$
Proof of Lemma 128

Applying the standard recursion for the B-spline basis functions at parameter $t_k$,

$$N_{i,k}(t_k) = \frac{t_k - t_i}{t_{i+k} - t_i} N_{i,k-1}(t_k) + \frac{t_{i+k+1} - t_k}{t_{i+k+1} - t_{i+1}} N_{i+1,k-1}(t_k),$$

for $i = 0$ (and subsequently for $i = j$ and $k - j$ for $j \in \{1, \ldots, k - 1\}$) yields

$$N_{0,k}(t_k) = \frac{t_k - t_0}{t_k - t_0} N_{0,k-1}(t_k) + \frac{t_{k+1} - t_k}{t_{k+1} - t_1} N_{1,k-1}(t_k)$$

$$= \frac{t_{k+1} - t_k}{t_{k+1} - t_k} N_{1,k-1}(t_k) = N_{1,k-1}(t_k)$$

$$= \frac{t_k - t_1}{t_k - t_1} N_{1,k-2}(t_k) + \frac{t_{k+1} - t_k}{t_{k+1} - t_2} N_{2,k-2}(t_k)$$

$$= N_{2,k-2}(t_k) = \cdots = N_{k,0}(t_k)$$

$$= 1.$$

Hence, due to the Partition of Unity, Cor. 125, $N_{i,k}(t_k) = 0$ for $i > 0$ and we get

$$\sum_{i=0}^{n} N_{i,k}(t_k) p_i = N_{0,k}(t_k) p_0 = p_0.$$

\[\square\]
Generation of Knot Vector

- Suppose that a B-spline curve over $[0, 1]$ has $n + 1$ control points $p_0, p_1, \ldots, p_n$ and degree $k$.
- We need $m + 1$ knots, where $m = n + k + 1$.
- If the B-spline curve is clamped then we get
  
  $$t_0 = t_1 = \ldots = t_k = 0 \quad \text{and} \quad t_{n+1} = t_{n+2} = \ldots = t_{n+k+1} = 1.$$

- The remaining $n - k$ knots can be spaced uniformly or non-uniformly.
- For uniformly spaced internal knots the interval $[0, 1]$ is divided into $n - k + 1$ subintervals. In this case the knots are given as follows:

  $$t_0 = t_1 = \ldots = t_k = 0$$

  $$t_{k+j} = \frac{j}{n - k + 1} \quad \text{for} \ j = 1, 2, \ldots, n - k$$

  $$t_{n+1} = t_{n+2} = \ldots = t_{n+k+1} = 1$$
Suppose that \( n = 6 \), i.e., that we have seven control points \( p_0, \ldots, p_6 \) and want to construct a clamped cubic B-spline curve. (Hence, \( k = 3 \).)

We have in total \( m + 1 = n + k + 2 = 6 + 3 + 2 = 11 \) knots and get

\[
\tau := (0, 0, 0, 0, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, 1, 1, 1, 1)
\]

as uniform knot vector.

For \( (p_0, \ldots, p_6) := (\left( \begin{array}{c} 0 \\ 0 \end{array} \right), \left( \begin{array}{c} 0.2 \\ 2 \end{array} \right), \left( \begin{array}{c} 2 \\ 3 \end{array} \right), \left( \begin{array}{c} 3 \\ 2 \end{array} \right), \left( \begin{array}{c} 3 \\ 0 \end{array} \right), \left( \begin{array}{c} 5 \\ 0 \end{array} \right), \left( \begin{array}{c} 7 \\ 1 \end{array} \right) \) \) we get the following clamped, cubic and \( C^2 \)-continuous B-spline curve:
Properties of B-Spline Curves

Lemma 129

The lower the degree of a B-spline curve, the closer it follows its control polygon.

Sketch of Proof: The lower the degree, the fewer control points contribute to $\mathcal{P}(t)$. For $k = 1$ it is simply the convex combination of pairs of control points.

- Clamped uniform B-spline of degree 10 for a control polygon with 14 vertices:

$$\{(1, 1), (1, 3), (3, 5), (5, 4), (6, 2), (3, 1), (3, 2), (11, 1), (8, 3), (8, 5), (10, 6), (4, 7), (1, 5)\}$$
Properties of B-Spline Curves

Lemma 130 (*Variation diminishing property*)

If a straight line intersects the control polygon of a B-spline curve \( m \) times then it intersects the actual B-spline curve at most \( m \) times.

Lemma 131 (*Affine invariance*)

Any B-spline representation is affinely invariant, i.e., given any affine map \( \pi \), the image curve \( \pi(P) \) of a B-spline curve \( P \) with control points \( p_0, p_1, \ldots, p_n \) has the control points \( \pi(p_0), \pi(p_1), \ldots, \pi(p_n) \).

*Sketch of Proof*: The proof is identical to the proof of the affine invariance of Bézier curves, recall Lem. 94.
Properties of B-Spline Curves

Lemma 132

Let $\mathcal{P}$ be a clamped B-spline curve of degree $k$ over $[0, 1]$ defined by $k + 1$ control points with position vectors $p_0, p_1, \ldots, p_k$ and the knot vector $\tau := (t_0, t_1, \ldots, t_{2k+1})$, for $k \in \mathbb{N}_0$. Then $\mathcal{P}$ is a Bézier curve of degree $k$.

- Note: This implies $0 = t_0 = t_1 = \ldots = t_k$ and $1 = t_{k+1} = \ldots = t_{2k} = t_{2k+1}$.
- Of course, this lemma can also be formulated for a parameter interval other than $[0, 1]$.
- Clamped (uniform) B-spline of degree 3 for knot vector $(0, 0, 0, 0, 1, 1, 1, 1)$ and control polygon $\left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 3 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 0 \end{pmatrix} \right\}$.
Derivatives of B-Spline Curves

Lemma 133

Let \( \mathcal{P} \) be a B-spline curve of degree \( k \) defined by \( n + 1 \) control points with position vectors \( p_0, p_1, \ldots, p_n \), and the knot vector \( \tau := (t_0, t_1, \ldots, t_{n+k+1}) \), for \( n \in \mathbb{N} \) and \( k \in \mathbb{N}_0 \) with \( k \leq n \). Then

\[
\mathcal{P}'(t) = \sum_{i=0}^{n-1} N_{i+1,k-1}(t) q_i \quad \text{for } t \in [t_k, t_{n+1}[,
\]

where

\[
q_i := \frac{k}{t_{i+k+1} - t_{i+1}} (p_{i+1} - p_i) \quad \text{for } i \in \{0, 1, \ldots, n - 1\}
\]

and the knot vector \( \tau \) remains unchanged.

Sketch of Proof: This is a consequence of Lem. 122 and some (lengthy) analysis.

Since the first derivative of a B-spline curve is another B-spline curve, one can apply this technique recursively to compute higher-order derivatives.
Derivatives of B-Spline Curves

Lemma 134

Let \( \mathcal{P} \) be a B-spline curve of degree \( k \) defined by \( n + 1 \) control points with position vectors \( p_0, p_1, \ldots, p_n \) and the clamped knot vector \( \tau := (t_0, t_1, \ldots, t_{n+k+1}) \), for \( n \in \mathbb{N} \) and \( k \in \mathbb{N}_0 \) with \( k \leq n \). Then, for the new knot vector \( \tau' := (t_1, t_2, \ldots, t_{n+k-1}, t_{n+k}) \),

\[
\mathcal{P}'(t) = \sum_{i=0}^{n-1} N_{i,k-1,\tau'}(t)q_i \quad \text{for } t \in [t_k, t_{n+1}],
\]

where

\[
q_i := \frac{k}{t_{i+k+1} - t_{i+1}}(p_{i+1} - p_i) \quad \text{for } i \in \{0, 1, \ldots, n-1\}.
\]

**Sketch of Proof:** One can show that \( N_{i+1,k-1,\tau}(t) \) is equal to \( N_{i,k-1,\tau'}(t) \) for all \( t \in [t_k, t_{n+1}] \). \( \square \)
Corollary 135

A clamped B-spline curve is tangent to the first leg and tangent to the last leg of its control polygon.

**Sketch of Proof:** Recall that, by Lem. 134, the first derivative of a clamped B-spline curve $\mathcal{P}$ of degree $k$ is a clamped B-spline curve of degree $k - 1$ over essentially the same knot vector but with new control points of the form

$$q_i := \frac{k}{t_{i+k+1} - t_{i+1}} (p_{i+1} - p_i) \quad \text{for } i \in \{0, 1, \ldots, n-1\}.$$ 

Hence, by arguments similar to those used in the proof of Lem. 128, one can show that $\mathcal{P}'(t_k)$ starts in $q_0$ and, thus, the tangent of $\mathcal{P}$ in the start point $\mathcal{P}(t_k)$ is parallel to $p_1 - p_0$.  

$\square$
Strong Convex Hull Property

Lemma 136 *(Strong convex hull property)*

Let \( P \) be a B-spline curve of degree \( k \) defined by \( n + 1 \) control points with position vectors \( p_0, p_1, \ldots, p_n \) and the knot vector \( \tau := (t_0, t_1, \ldots, t_{n+k+1}) \), for \( n \in \mathbb{N} \) and \( k \in \mathbb{N}_0 \) with \( k \leq n \). For \( i \in \mathbb{N} \) with \( k \leq i \leq n \), we have

\[
P|_{[t_i, t_{i+1}]} \subset \text{CH}(\{p_{i-k}, p_{i-k+1}, \ldots, p_{i-1}, p_i\}).
\]

**Proof:** Lemma 120 tells us that \( N_{i-k,k}, N_{i-k+1,k}, \ldots, N_{i-1,k}, N_{i,k} \) are the only B-spline basis functions that can be non-zero over \([t_i, t_{i+1}]\), for \( k \leq i \leq n \), while all other basis functions are zero (Lem. 118). Together with Cor. 125, Partition of Unity, we get

\[
1 = \sum_{j=0}^{n} N_{j,k}(t) = \sum_{j=i-k}^{i} N_{j,k}(t) \quad \text{for all } t \in [t_i, t_{i+1}].
\]

Hence,

\[
P(t) = \sum_{j=0}^{n} N_{j,k}(t)p_j = \sum_{j=i-k}^{i} N_{j,k}(t)p_j \quad \text{for all } t \in [t_i, t_{i+1}]
\]

is a convex combination of \( p_{i-k}, p_{i-k+1}, \ldots, p_{i-1}, p_i \).
**Strong Convex Hull Property**

**Lemma 136 (Strong convex hull property)**

Let \( \mathcal{P} \) be a B-spline curve of degree \( k \) defined by \( n + 1 \) control points with position vectors \( p_0, p_1, \ldots, p_n \) and the knot vector \( \tau := (t_0, t_1, \ldots, t_{n+k+1}) \), for \( n \in \mathbb{N} \) and \( k \in \mathbb{N}_0 \) with \( k \leq n \). For \( i \in \mathbb{N} \) with \( k \leq i \leq n \), we have

\[
\mathcal{P}|_{[t_i, t_{i+1}] \subset \text{CH} \left( \{p_{i-k}, p_{i-k+1}, \ldots, p_{i-1}, p_i\} \right)}.
\]

Second knot span of a cubic B-spline contained in \( \text{CH}(\{p_1, p_2, p_3, p_4\}) \).
Local Control and Modification

Lemma 137 (Local control)

Let \( \mathcal{P} \) be a B-spline curve of degree \( k \) defined by \( n + 1 \) control points with position vectors \( p_0, p_1, \ldots, p_n \) and the knot vector \( \tau := (t_0, t_1, \ldots, t_{n+k+1}) \), for \( n \in \mathbb{N} \) and \( k \in \mathbb{N}_0 \) with \( k \leq n \). Then the B-spline curve \( \mathcal{P} \) restricted to \([t_i, t_{i+1}]\) depends only on the positions of \( p_{i-k}, p_{i-k+1}, \ldots, p_{i-1}, p_i \).

Proof: By Lem. 120, and as in the proof of Lem. 136,

\[
\mathcal{P}_{|[t_i, t_{i+1}]}(t) = \sum_{j=i-k}^{i} N_{j,k}(t) p_j.
\]

Lemma 138 (Local modification scheme)

Let \( \mathcal{P} \) be a B-spline curve of degree \( k \) defined by \( n + 1 \) control points with position vectors \( p_0, p_1, \ldots, p_n \) and the knot vector \( \tau := (t_0, t_1, \ldots, t_{n+k+1}) \), for \( n \in \mathbb{N} \) and \( k \in \mathbb{N}_0 \) with \( k \leq n \). Then a modification of the position of \( p_i \) changes \( \mathcal{P} \) only in the parameter interval \([t_i, t_{i+k+1}],[ for i \in \{0, 1, \ldots, n\}.

Proof: The Local Support Lemma 117 tells us that

\[
N_{i,k}(t) = 0 \quad \text{if} \quad t \notin [t_i, t_{i+k+1}].
\]
Local Control and Modification

- Clamped uniform B-spline of degree three with knot vector
  \[ \tau := (0, 0, 0, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 11, 11, 11) \]
  for a control polygon with 14 vertices:
  \[ \{(1, 1), (1, 3), (3, 5), (5, 4), (5, 2), (3, 1), (3, 11), (8, 8), (8, 10), (4, 6), (4, 7), (1, 5)\} \]
Local Control and Modification

- Clamped uniform B-spline of degree three with knot vector
  \[ \tau := (0, 0, 0, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 11, 11, 11) \]

  for a control polygon with 14 vertices:

  \[ \left\{ \left( \begin{array}{c} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{array} \right), \left( \begin{array}{c} 1 \\ 3 \\ 3 \\ 5 \\ 4 \\ 2 \\ 2 \\ 1 \\ 1 \\ 1 \\ 3 \\ 5 \end{array} \right), \left( \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array} \right) \right\} \]
Multiple Control Points

**Lemma 139 (Multiple control points)**

Let \( \mathcal{P} \) be a B-spline curve of degree \( k \) defined by \( n + 1 \) control points with position vectors \( p_0, p_1, \ldots, p_n \) and the knot vector \( \tau := (t_0, t_1, \ldots, t_{n+k+1}) \), for \( n \in \mathbb{N} \) and \( k \in \mathbb{N}_0 \) with \( k \leq n \).

1. If \( k \) control points \( p_{i-k+1}, p_{i-k+2}, \ldots, p_i \) coincide, i.e., if \( p_{i-k+1} = p_{i-k+2} = \ldots = p_i \) then \( \mathcal{P} \) contains \( p_i \) and is tangent to the legs \( p_{i-k}p_{i-k+1} \) and \( p_ip_{i+1} \) of the control polygon, for \( i \in \mathbb{N} \) with \( k \leq i < n \).

2. If \( k \) control points \( p_{i-k+1}, p_{i-k+2}, \ldots, p_i \) are collinear then \( \mathcal{P} \) touches a leg of the control polygon, for \( i \in \mathbb{N} \) with \( k \leq i < n \).

3. If \( k + 1 \) control points \( p_{i-k}, p_{i-k+1}, \ldots, p_i \) are collinear then \( \mathcal{P} \) coincides with a leg of the control polygon, for \( i \in \mathbb{N} \) with \( k < i < n \).

**Sketch of Proof:** This is a consequence of Lemma 120 and of the Strong Convex Hull Property (Lem. 136).

\[\square\]

- Note that this implies that a degree-\( k \) B-spline \( \mathcal{P} \) starts at \( p_0 \) if \( p_0 = p_1 = \ldots = p_{k-1} \).
Multiple Control Points

Clamped cubic B-spline with control points

\[
\begin{pmatrix}
2 \\
0
\end{pmatrix}, \begin{pmatrix}
0 \\
4
\end{pmatrix}, \begin{pmatrix}
4 \\
5
\end{pmatrix}, \begin{pmatrix}
8 \\
4
\end{pmatrix}, \begin{pmatrix}
8 \\
2
\end{pmatrix}, \begin{pmatrix}
6 \\
0
\end{pmatrix}
\]

and uniform knot vector \( (0, 0, 0, 0, 1, 2, 3, 3, 3, 3) \):
Multiple Control Points

Clamped cubic B-spline with control points

\[
\begin{pmatrix}
2 \\ 0 \\
0 \\ 4 \\
4 \\ 5 \\
4 \\ 5 \\
4 \\ 5 \\
8 \\ 4 \\
8 \\ 2 \\
4 \\ 6 \\
0 \\
\end{pmatrix},
\]

and uniform knot vector \((0, 0, 0, 0, 1, 2, 3, 4, 5, 5, 5, 5)\):
Multiple Control Points

Clamped cubic B-spline with control points

\[
\begin{pmatrix} 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 4 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \end{pmatrix}, \begin{pmatrix} 6 \\rac{9}{2} \end{pmatrix}, \begin{pmatrix} 8 \\ 4 \end{pmatrix}, \begin{pmatrix} 8 \\ 2 \end{pmatrix}, \begin{pmatrix} 6 \\ 0 \end{pmatrix}
\]

and uniform knot vector \((0, 0, 0, 0, 1, 2, 3, 4, 4, 4, 4)\):
Multiple Control Points

Clamped cubic B-spline with control points

\[
\begin{pmatrix}
2 \\
0
\end{pmatrix}, \begin{pmatrix}
0 \\
4
\end{pmatrix}, \begin{pmatrix}
4 \\
5
\end{pmatrix}, \begin{pmatrix}
5 \\
\frac{19}{4}
\end{pmatrix}, \begin{pmatrix}
7 \\
\frac{17}{4}
\end{pmatrix}, \begin{pmatrix}
8 \\
8
\end{pmatrix}, \begin{pmatrix}
8 \\
4
\end{pmatrix}, \begin{pmatrix}
6 \\
2
\end{pmatrix}, \begin{pmatrix}
0
\end{pmatrix}
\]

and uniform knot vector \((0, 0, 0, 0, 1, 2, 3, 4, 5, 5, 5, 5)\):
Lemma 140 (*Multiple knots*)

Let $\mathcal{P}$ be a B-spline curve of degree $k$ defined by $n + 1$ control points with position vectors $p_0, p_1, \ldots, p_n$ and the knot vector $\tau := (t_0, t_1, \ldots, t_{n+k+1})$, for $n \in \mathbb{N}$ and $k \in \mathbb{N}_0$ with $k \leq n$. Let $i \in \mathbb{N}$ with $k + 1 \leq i \leq n - k$. If $t_i$ is a knot of multiplicity $k$, i.e., if $t_i = t_{i+1} = \ldots = t_{i+k-1}$ then $\mathcal{P}(t_i) = p_{i-1}$ and $\mathcal{P}$ is tangent to the legs $\overline{p_{i-2}p_{i-1}}$ and $\overline{p_{i-1}p_i}$ of the control polygon.
Multiple Knots

- Clamped uniform B-spline of degree three for a control polygon with nine vertices:

\[
\begin{bmatrix}
-1 & 0 & 0 \\
0 & 2 & 3 \\
0 & 4 & 6 \\
2 & 0 & 8 \\
4 & 3 & 8 \\
6 & 2 & 0 \\
8 & 0 & 9
\end{bmatrix}
\]

Knot vector:

\[\tau := (0, 0, 0, 0, 1, 2, 3, 4, 5, 6, 6, 6, 6, 6)\]
Clamped uniform B-spline of degree three for a control polygon with nine vertices:

\[
\left\{ \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 6 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 8 \\ 8 \\ 0 \end{pmatrix}, \begin{pmatrix} 9 \\ 8 \\ 0 \end{pmatrix} \right\}
\]

Knot vector:

\[ \tau := (0, 0, 0, 0, 1, 2, 2, 2, 3, 4, 4, 4, 4) \]
Motivation for de Boor’s Algorithm

- Can we express $P(t)$ in terms of $N_{i,0}(t)$?
- We exploit the recursive definition of $N_{i,k}$ in order to determine $P(t)$ in terms of $N_{i,k-1}(t)$, recalling that $t \in [t_k, t_{n+1}]$.

\[
P(t) = \sum_{i=0}^{n} N_{i,k}(t) p_i = \sum_{i=0}^{n} \left( \frac{t - t_i}{t_{i+k} - t_i} N_{i,k-1}(t) + \frac{t_{i+k+1} - t}{t_{i+k+1} - t_{i+1}} N_{i+1,k-1}(t) \right) p_i
\]

\[
= \sum_{i=0}^{n} \frac{t - t_i}{t_{i+k} - t_i} N_{i,k-1}(t) p_i + \sum_{i=0}^{n} \frac{t_{i+k+1} - t}{t_{i+k+1} - t_{i+1}} N_{i+1,k-1}(t) p_i
\]

\[
= \frac{t - t_0}{t_k - t_0} N_{0,k-1}(t) p_0 + \sum_{i=1}^{n} \frac{t - t_i}{t_{i+k} - t_i} N_{i,k-1}(t) p_i + \frac{t_{n+k+1} - t}{t_{n+k+1} - t_{n+1}} N_{n+1,k-1}(t) p_n + \sum_{i=0}^{n-1} \frac{t_{i+k+1} - t}{t_{i+k+1} - t_{i+1}} N_{i+1,k-1}(t) p_i
\]

\[
= \sum_{i=1}^{n} \frac{t - t_i}{t_{i+k} - t_i} N_{i,k-1}(t) p_i + \sum_{i=1}^{n} \frac{t_{i+k} - t}{t_{i+k} - t_i} N_{i,k-1}(t) p_{i-1}
\]

\[
= \sum_{i=1}^{n} N_{i,k-1}(t) \left( \frac{t_{i+k} - t}{t_{i+k} - t_i} p_{i-1} + \frac{t - t_i}{t_{i+k} - t_i} p_i \right) =: \sum_{i=1}^{n} N_{i,k-1}(t) p_{i,1}(t)
\]
Motivation for de Boor’s Algorithm

Equality at * holds since each basis function $N_{i,k}$ is non-zero only over $[t_i, t_{i+k+1}]$ (Local Support Lem. 117):

\[ N_{0,k-1,t}(t) = 0 \text{ for } t \not\in [t_0, t_k] \quad \text{and} \quad N_{n+1,k-1,t}(t) = 0 \text{ for } t \not\in [t_{n+1}, t_{n+k+1}] \]

For $1 \leq i \leq n$, we have

\[ p_{i,1}(t) := (1 - \alpha_{i,1}) p_{i-1} + \alpha_{i,1} p_i \quad \text{with} \quad \alpha_{i,1} := \frac{t - t_i}{t_{i+k} - t_i}, \]

thus expressing $P(t)$ in terms of basis functions of degree $k - 1$ and modified (parameter-dependent!) new control points.

Repeating this process yields

\[ P(t) = \sum_{i=2}^{n} N_{i,k-2,t}(t)p_{i,2}(t), \]

where, for $2 \leq i \leq n$,

\[ p_{i,2}(t) := (1 - \alpha_{i,2}) p_{i-1,1}(t) + \alpha_{i,2} p_{i,1}(t) \quad \text{with} \quad \alpha_{i,2} := \frac{t - t_i}{t_{i+k-1} - t_i}, \]
De Boor’s Algorithm

Theorem 141 (de Boor’s algorithm)

Let $\mathcal{P}$ be a B-spline curve of degree $k$ with control points $p_0, p_1, \ldots, p_n$ and knot vector $\tau := (t_0, t_1, \ldots, t_{n+k+1})$. If we define

$$p_{i,j}(t) := \begin{cases} p_i & \text{if } j = 0, \\ (1 - \alpha_{i,j}) \, p_{i-1,j-1}(t) + \alpha_{i,j} \, p_{i,j-1}(t) & \text{if } j > 0, \end{cases}$$

where

$$\alpha_{i,j} := \frac{t - t_i}{t_{i+k+1-j} - t_i},$$

then

$$\mathcal{P}(t) = \sum_{i=k}^{n} N_{i,0}(t) p_{i,k}(t) \quad \text{for } t \in [t_k, t_{n+1}].$$

Corollary 142

Let $\mathcal{P}$ be a B-spline curve of degree $k$ with control points $p_0, p_1, \ldots, p_n$ and knot vector $\tau := (t_0, t_1, \ldots, t_{n+k+1})$. If $t \in [t_i, t_{i+1}]$, for $i \in \{k, k+1, \ldots, n\}$, then $\mathcal{P}(t) = p_{i,k}(t)$. 
Sample Run of de Boor’s Algorithm

- Clamped uniform B-spline of degree three for seven control points:

\[
\{(0, 0), (2, 3), (4, 0), (2, 3), (8, 2), (8, 0)\}
\]

Knot vector with eleven knots: \(\tau := (0, 0, 0, 0, 1, 2, 3, 4, 4, 4, 4)\).

- B-spline curve with \(p_{i,0}(0.7)\), with \(0.7 \in [t_3, t_4[\):

\[
\{(0, 0), (2, 3), (4, 0), (2, 3), (8, 2), (8, 0)\}
\]
Sample Run of de Boor’s Algorithm

- Clamped uniform B-spline of degree three for seven control points:

\[
\left\{ \left( \begin{array}{c} 0 \\ 0 \\ 2 \\ 3 \\ 4 \\ 6 \\ 8 \\ 8 \\ 0 \end{array} \right) \right\}
\]

Knot vector with eleven knots: \( \tau := (0, 0, 0, 0, 1, 2, 3, 4, 4, 4) \).

- B-spline curve with \( p_{i,1}(0.7) \), with \( 0.7 \in [t_3, t_4] \):

\[
\left\{ \left( \begin{array}{c} 0.1 \\ 0.7 \\ 2.4667 \\ 2.35 \\ 2.3 \end{array} \right) \right\}
\]
Sample Run of de Boor’s Algorithm

- Clamped uniform B-spline of degree three for seven control points:

\[
\{ (0, 0), (0, 2), (2, 3), (4, 0), (6, 3), (8, 2), (8, 0) \}
\]

Knot vector with eleven knots: \( \tau := (0, 0, 0, 0, 1, 2, 3, 4, 4, 4, 4) \).

- B-spline curve with \( p_{i,2}(0.7) \), with \( 0.7 \in [t_3, t_4] \):

\[
\{ (0.49, 2.065), (1.3183, 2.3325) \}
\]
Sample Run of de Boor’s Algorithm

- Clamped uniform B-spline of degree three for seven control points:

\[
\left\{ \left( \begin{array}{c} 0 \\ 0 \end{array} \right), \left( \begin{array}{c} 0 \\ 2 \end{array} \right), \left( \begin{array}{c} 2 \\ 3 \end{array} \right), \left( \begin{array}{c} 4 \\ 0 \end{array} \right), \left( \begin{array}{c} 6 \\ 3 \end{array} \right), \left( \begin{array}{c} 8 \\ 2 \end{array} \right), \left( \begin{array}{c} 8 \\ 0 \end{array} \right) \right\}
\]

Knot vector with eleven knots: \( \tau := (0, 0, 0, 0, 1, 2, 3, 4, 4, 4) \).

- B-spline curve with \( p_{i,3}(0.7) \), with \( 0.7 \in [t_3, t_4] \):

\[
\left\{ \left( \begin{array}{c} 1.0698 \\ 2.2523 \end{array} \right) \right\} = \{ p(0.7) \}\]
De Boor’s Algorithm for Subdividing a B-Spline Curve

- Clamped uniform B-spline of degree three for seven control points:
  \[
  \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 2 \\ 2 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 3 \\ 3 \\ 2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 4 \\ 0 \\ 6 \\ 2 \\ 0 \\ 8 \end{pmatrix}, \begin{pmatrix} 0 \\ 4 \\ 8 \\ 8 \\ 8 \end{pmatrix} \right\}
  \]

Knot vector with eleven knots: \( \tau := (0, 0, 0, 0, 1, 2, 3, 4, 4, 4, 4) \).

- New control polygons for \( t^* := 0.7 \):
  \[
  (p_{0,0}, p_{1,1}, p_{2,2}, p_{3,3}) \quad \text{and} \quad (p_{3,3}, p_{3,2}, p_{3,1}, p_{3,0}, p_{4,0}, p_{5,0}, p_{6,0})
  \]

- New knot vectors for \( t^* := 0.7 \):
  \[
  (0, 0, 0, 0, 0.7, 0.7, 0.7, 0.7) \quad \text{and} \quad (0.7, 0.7, 0.7, 0.7, 1, 2, 3, 4, 4, 4, 4)
  \]
De Boor’s Algorithm for Subdividing a B-Spline Curve

Definition 143

Let \( \mathcal{P} \) be a clamped B-spline curve of degree \( k \) defined by \( n + 1 \) control points with position vectors \( p_0, p_1, \ldots, p_n \) and the clamped knot vector \( \tau := (t_0, t_1, \ldots, t_{n+k+1}) \), for \( n \in \mathbb{N} \) and \( k \in \mathbb{N}_0 \) with \( k \leq n \). For some \( t^* \in [t_i, t_{i+1}], \) with \( i \in \{k, \ldots, n\} \), we define two new knot vectors \( \tau^*, \tau^{**} \) and two new control polygons \( P^*, P^{**} \) as follows:

- If \( t^* \neq t_i \) then \( m := i \) else \( m := i - 1 \).
- \( \tau^* := (t_0, t_1, \ldots, t_m, t^*, \ldots, t^*) \) and \( \tau^{**} := (t^*, \ldots, t^*, t_{m+1}, \ldots, t_{n+k+1}) \), \((k+1)\) times \((k+1)\) times

- \( P^*(t^*) := (p_{0,0}(t^*), p_{1,0}(t^*), \ldots, p_{m-k,0}(t^*), p_{1,1}(t^*), p_{2,2}(t^*), \ldots, p_{k,k}(t^*)) \),

- \( P^{**}(t^*) := (p_{k,k}(t^*), p_{k,k-1}(t^*), \ldots, p_{k,1}(t^*), p_{m,0}(t^*), p_{m+1,0}(t^*), \ldots, p_{n,0}(t^*)) \),

where the new control points \( p_{i,j}(t^*) \) are obtained by de Boor’s algorithm (Thm 143).
De Boor’s Algorithm for Subdividing a B-Spline Curve

Lemma 144

Let \( \mathcal{P} \) be a clamped B-spline curve of degree \( k \) defined by \( n + 1 \) control points with position vectors \( p_0, p_1, \ldots, p_n \) and the clamped knot vector \( \tau := (t_0, t_1, \ldots, t_{n+k+1}) \), for \( n \in \mathbb{N} \) and \( k \in \mathbb{N}_0 \) with \( k \leq n \). (Hence, \( t_0 = t_1 = \ldots = t_k < t_{k+1} \) and \( t_n < t_{n+1} = \ldots = t_{n+k+1} \).) For some \( t^* \in [t_i, t_{i+1}[ \), with \( i \in \{k, \ldots, n\} \), we define two new knot vectors \( \tau^*, \tau^{**} \) and two new control polygons \( \mathcal{P}^*, \mathcal{P}^{**} \) as in Def. 143. Then we get two new B-spline curves \( \mathcal{P}^* \) and \( \mathcal{P}^{**} \) of degree \( k \) with control polygon \( \mathcal{P}^* \) (\( \mathcal{P}^{**} \), resp.) and knot vector \( \tau^* \) (\( \tau^{**} \), resp.) that join in a tangent-continuous way at point \( p_{kk}(t^*) = \mathcal{P}(t^*) \), such that

\[
\mathcal{P}^* = \mathcal{P}|_{[t_k, t^*[} \quad \text{and} \quad \mathcal{P}^{**} = \mathcal{P}|_{[t^*, t_{n+1}[}. \]

De Boor’s Algorithm for Splitting a B-Spline Curve into Bézier Segments

Corollary 145

Let $\mathcal{P}$ be a clamped B-spline curve of degree $k$ defined by $n + 1$ control points with position vectors $p_0, p_1, \ldots, p_n$ and the clamped knot vector $\tau := (t_0, t_1, \ldots, t_{n+k+1})$, for $n \in \mathbb{N}$ and $k \in \mathbb{N}_0$ with $k \leq n$. (Hence, $t_0 = t_1 = \ldots = t_k < t_{k+1}$ and $t_n < t_{n+1} = \ldots = t_{n+k+1}$. ) Subdividing $\mathcal{P}$ at the knot values $\{t_{k+1}, t_{k+2}, \ldots, t_{n-1}, t_n\}$, as outlined in Def. 143, splits $\mathcal{P}$ into $n - k + 1$ Bézier curves of degree $k$.

Sketch of Proof: Lemma 144 ensures that each of the resulting curves is a B-spline curve of degree $k$, where the $m$-th curve is defined over $[t_{k+m}, t_{k+m+1}]$, for $m \in \{0, 1, \ldots, n - k\}$. Each curve has knot vectors of length $2k + 2$, with start and end knots of multiplicity $k + 1$ but no interior knots. After mapping $[t_{k+m}, t_{k+m+1}]$ to $[0, 1]$ we can apply Lem. 132 and conclude that the resulting B-spline curve is a Bézier curve of degree $k$. \qed
De Boor’s Algorithm for Splitting a B-Spline Curve into Bézier Segments

- Clamped uniform B-spline of degree three for seven control points:
  \[
  \left\{ \left( \begin{array}{c} 0 \\ 0 \\ 2 \\ 3 \\ 4 \\ 6 \\ 8 \\ 8 \\ 8 \\ 0 \end{array} \right) \right\}
  \]

  Knot vector with eleven knots: \( \tau := (0, 0, 0, 0, 1, 2, 3, 4, 4, 4, 4) \).

- Third Bézier segment over \([2, 3]\).

- Note that the number of knots increased drastically!
Knot Insertion

▶ Suppose that we would like to insert a new knot $t^* \in [t_j, t_{j+1}[$, for some $j \in \{k, k + 1, \ldots, n\}$, into the knot vector

$$\tau := (t_0, t_1, \ldots, t_j, t_{j+1}, \ldots, t_{n+k+1}),$$

thus transforming $\tau$ into a knot vector

$$\tau^* := (t_0, t_1, \ldots, t_j, t^*, t_{j+1}, \ldots, t_{n+k+1}).$$

▶ The fundamental equality $m = n + k + 1$, with $m + 1$ denoting the number of knots, tells us that we will have to either increase the number $n$ of control points by one or to increase the degree $k$ of the curve by one.

▶ Since an increase of the degree would change the shape of the B-spline globally, we opt for increasing the number of control points (and modifying some of them).

▶ How can we modify the control points such that the shape of the curve is preserved?
Knot Insertion

Lemma 146 (*Boehm 1980*)

Let $\mathcal{P}$ be a B-spline curve of degree $k$ defined by $n + 1$ control points with position vectors $p_0, p_1, \ldots, p_n$ and the knot vector $\tau := (t_0, t_1, \ldots, t_{n+k+1})$, for $n \in \mathbb{N}$ and $k \in \mathbb{N}_0$ with $k \leq n$. Let $t^* \in [t_j, t_{j+1}[$, for some $j \in \{k, k + 1, \ldots, n\}$, and define a knot vector $\tau^*$ as $\tau^* := (t_0, t_1, \ldots, t_j, t^*, t_{j+1}, \ldots, t_{n+k+1})$. Then we have

$$\mathcal{P}(t) = \sum_{i=0}^{n} N_{i,k,\tau}(t)p_i = \sum_{i=0}^{n+1} N_{i,k,\tau^*}(t)p_i^* =: \mathcal{P}^*(t) \quad \text{for all } t \in [t_k, t_{n+1}[$$

if, for $0 \leq i \leq n + 1$,

$$p_i^* := \begin{cases} p_i & \text{if } i \leq j - k \\ (1 - \alpha_i)p_{i-1} + \alpha_ip_i & \text{if } j - k + 1 \leq i \leq j \\ p_{i-1} & \text{if } i \geq j + 1 \end{cases}$$

and

$$\alpha_i := \frac{t^* - t_i}{t_{i+k} - t_i} \quad \text{for } i \in \{j - k + 1, \ldots, j\}.$$
Knot Insertion: Sample

- Clamped uniform B-spline of degree three for 14 control points and knot vector with 18 knots: \( \tau := (0, 0, 0, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 11, 11, 11) \):

\[
\{(1), (1), (3), (3), (5), (5), (1), (6), (2), (2), (3), (3), (11), (11), (8), (8), (6), (10), (4), (4), (1)\}
\]

- New 15 control points for 19 knots

\( \tau^* := (0, 0, 0, 0, 1, 2, 3, 4, 4.2, 5, 6, 7, 8, 9, 10, 11, 11, 11, 11) \):

\[
\{(1), (1), (3), (3), (5), (5), (6), (2.5333), (4.2), (1.9333), (3), (3), (11), (11), (8), (8), (6), (10), (4), (1)\}
\]
Knot Insertion: Sample

Old 18 knots:

\[ t = 4.2 \]

\[ t^* = 4.2 \]

For \( t^* := 4.2 = 4\frac{1}{5} \) we have \( j = 7 \) and \( j - k + 1 = 5 \), and get:

\[ \alpha_5 := \frac{t^* - t_5}{t_8 - t_5} = \frac{2\frac{1}{5}}{3} = \frac{11}{15} \]

\[ p_5^* = (1 - \alpha_5)p_4 + \alpha_5p_5 = \frac{1}{15}\left(\begin{array}{c}79 \\ 38\end{array}\right) \approx \left(\begin{array}{c}5.2667 \\ 2.5333\end{array}\right) \]

\[ \alpha_6 := \frac{t^* - t_6}{t_9 - t_6} = \frac{1\frac{1}{5}}{3} = \frac{6}{15} \]

\[ p_6^* = (1 - \alpha_6)p_5 + \alpha_6p_6 = \frac{1}{15}\left(\begin{array}{c}63 \\ 30\end{array}\right) \approx \left(\begin{array}{c}4.2 \\ 2\end{array}\right) \]

\[ \alpha_7 := \frac{t^* - t_7}{t_10 - t_7} = \frac{\frac{1}{5}}{3} = \frac{1}{15} \]

\[ p_7^* = (1 - \alpha_7)p_6 + \alpha_7p_7 = \frac{1}{15}\left(\begin{array}{c}45 \\ 29\end{array}\right) \approx \left(\begin{array}{c}3 \\ 1.9333\end{array}\right) \]
Knot Insertion and Deletion

- The so-called Oslo algorithm, developed by Cohen et al. [1980], is more general than Boehm’s algorithm: It allows the insertion of several (possibly multiple) knots into a knot vector. (It is also substantially more complex, though.)

- An algorithm for the removal of a knot is due to Tiller [1992]. However, as pointed out by Tiller, knot removal and degree reduction result in an overspecified problem which, in general, can only be solved within some tolerance.
Closed B-Spline Curves

Lemma 147

Let \( \mathcal{P} \) be a B-spline curve of degree \( k \) defined by \( n + 1 \) control points with position vectors \( p_0, p_1, \ldots, p_n \) and the uniform knot vector \( \tau := (t_0, t_1, \ldots, t_{n+k+1}) \), for \( n \in \mathbb{N} \) and \( k \in \mathbb{N}_0 \) with \( k \leq n \). If

\[
p_0 = p_{n-k+1}, \quad p_1 = p_{n-k+2}, \ldots, p_{k-2} = p_{n-1}, \quad p_{k-1} = p_n
\]

then \( \mathcal{P} \) is \( C^{k-1} \) at the joining point \( \mathcal{P}(t_k) = \mathcal{P}(t_{n+1}) \).

Lemma 148

Let \( \mathcal{P} \) be a B-spline curve of degree \( k \) defined by \( n + 2 \) control points with position vectors \( p_0, p_1, \ldots, p_n, p_{n+1} \) and the (possibly non-uniform and periodic) knot vector \( \tau := (t_0, t_1, \ldots, t_{n+k+1}, t_{n+k+2}) \), for \( n \in \mathbb{N} \) and \( k \in \mathbb{N}_0 \) with \( k \leq n \). If

\[
p_0 = p_{n+1} \quad \text{and} \quad t_0 = t_{n+1}, \quad t_1 = t_{n+2}, \ldots, t_k = t_{n+k+1}, \quad t_{k+1} = t_{n+k+2}
\]

then \( \mathcal{P} \) is \( C^{k-1} \) at the joining point \( \mathcal{P}(t_0) = \mathcal{P}(t_{n+1}) \).

\( \blacktriangleright \) Hence, wrapping around \( k \) control points or \( k + 2 \) knots achieves \( C^{k-1} \)-continuity at the joining point.
Closed B-Spline Curves via Wrapping of Control Points

- Uniform B-spline of degree three for nine control points:
\[
\{ \begin{pmatrix} 4 \\ 3 \\ 0 \\ 0 \\ 0 \\ 7 \\ 7 \\ 2 \\ 3 \\ 4 \\ 0 \\ 0 \end{pmatrix} \}
\]

Knot vector with 13 knots: \( \tau := (0, \frac{1}{12}, \frac{2}{12}, \frac{3}{12}, \frac{4}{12}, \frac{5}{12}, \frac{6}{12}, \frac{7}{12}, \frac{8}{12}, \frac{9}{12}, \frac{10}{12}, \frac{11}{12}, 1) \).
B-Spline Surfaces

Definition 149 (B-spline surface)

For \( n, m \in \mathbb{N} \) and \( k', k'' \in \mathbb{N}_0 \) with \( k' \leq n \) and \( k'' \leq m \), consider a set of \( (n+1) \times (m+1) \) control points with position vectors \( p_{i,j} \in \mathbb{R}^3 \) for \( 0 \leq i \leq n \) and \( 0 \leq j \leq m \), and let \( \sigma := (s_0, s_1, \ldots, s_{n+k'+1}) \) and \( \tau := (t_0, t_1, \ldots, t_{m+k''+1}) \) be two knot vectors. Then the B-spline surface relative to \( \sigma \) and \( \tau \) with control net \((p_{i,j})_{i,j=0}^{n,m}\) is given by

\[
S(s, t) := \sum_{i=0}^{n} \sum_{j=0}^{m} N_{i,k',\sigma}(s) N_{j,k'',\tau}(t) p_{i,j} \quad \text{for} \quad s \in [s_{k'}, s_{n+1}], t \in [t_{k''}, t_{m+1}],
\]

where \( N_{i,k',\sigma} \) is the \( i \)-th B-spline basis function of degree \( k' \) relative to \( \sigma \), and \( N_{j,k'',\tau} \) is the \( j \)-th B-spline basis function of degree \( k'' \) relative to \( \tau \).

\[ \blacktriangleright \quad \text{Hence, a B-spline surface is another example of a tensor-product surface.} \]
Sample B-Spline Surfaces
Properties of B-Spline Surfaces

Lemma 150 (Non-negativity)

With the setting of Def. 149, we have $N_{i,k',\sigma}(s)N_{j,k'',\tau}(t) \geq 0$ for all (permissible) $i, j \in \mathbb{Z}$ and $k', k'' \in \mathbb{N}_0$, and all real $s, t$.

Lemma 151 (Partition of unity)

With the setting of Def. 149, we have

$$\sum_{i=0}^{n} \sum_{j=0}^{m} N_{i,k',\sigma}(s)N_{j,k'',\tau}(t) = 1$$

for all $s \in [s_{k'}, s_{n+1}]$, $t \in [t_{k''}, t_{m+1}]$.

Lemma 152 (Strong convex hull property)

With the setting of Def. 149, for $i, j \in \mathbb{N}$ with $k' \leq i \leq n$ and $k'' \leq j \leq m$ we have

$$S|_{[s_i, s_{i+1}] \times [t_j, t_{j+1}]} \subset \text{CH}(\{p_{l',l'':} : i - k' \leq l' \leq i \land j - k'' \leq l'' \leq j\})$$
Properties of B-Spline Surfaces

Lemma 153 (*Local control*)

With the setting of Def. 149, for \( i, j \in \mathbb{N} \) with \( k' \leq i \leq n \) and \( k'' \leq j \leq m \) we have that

\[
S|_{[s_i, s_{i+1}] \times [t_j, t_{j+1}]} \text{ depends only on } \{p_{l', l''} : i - k' \leq l' \leq i \wedge j - k'' \leq l'' \leq j\}.
\]

Lemma 154 (*Local modification scheme*)

With the setting of Def. 149, a modification of the position of \( p_{i,j} \) changes \( S \) only in the parameter domain \([s_i, s_{i+k'+1}] \times [t_j, t_{j+k''+1}]\), for \( i \in \{0, 1, \ldots, n\} \) and \( j \in \{0, 1, \ldots, m\} \).

Lemma 155 (*Affine invariance*)

Any B-spline representation is affinely invariant, i.e., given any affine map \( \pi \), the image surface \( \pi(S) \) of a B-spline surface \( S \) with control points \( p_{i,j} \) has the control points \( \pi(p_{i,j}) \).
Clamping of a B-Spline Surface

- A B-spline surface can be clamped by repeating the same knot values in one direction of the parameters (i.e., in $s$ or $t$).
- We can also close the surface by recycling the control points.
- If a B-spline surface is closed in one direction, then the surface becomes a tube.
- If a B-spline surface is closed in two directions, then the surface becomes a torus.
- Other topologies are more difficult to handle, such as a ball or a double torus.
Evaluation of a B-Spline Surface

▶ Five easy steps to calculate a point on a B-spline patch for \((s, t)\)

1. Find the knot span in which \(s\) lies, i.e., find \(i\) such that \(s \in [s_i, s_{i+1}]\).
2. Evaluate the non-zero basis functions \(N_{i-k',k'}(s), \ldots, N_{i,k'}(s)\).
3. Find the knot span in which \(t\) lies, i.e., find \(j\) such that \(t \in [t_j, t_{j+1}]\).
4. Evaluate the non-zero basis functions \(N_{j-k'',k''}(t), \ldots, N_{j,k''}(t)\).
5. Multiply \(N_{i',k'}(s)\) with \(N_{j',k''}(t)\) and with the control point \(p_{i',j'}\), for \(i' \in \{i - k', \ldots, i\}\) and \(j' \in \{j - k'', \ldots, j\}\).

▶ Alternatively, one can apply an appropriate generalization of de Boor's algorithm.
Motivation

- Can we use a B-spline curve to represent a circular arc?

\[ \left\{ \begin{array}{cccc}
(1) & (1) & (0) & (-1) \\
(1) & (0) & (-1) & (-1) \\
(0) & (-1) & (1) & (0)
\end{array} \right\} \]

- uniform knots, degree 10
- close to a circle, but still no circle!
Non-Uniform Rational B-Splines

Definition 156 *(NURBS curve)*

For \( n \in \mathbb{N} \) and \( k \in \mathbb{N}_0 \) with \( k \leq n \), consider a set of \( n + 1 \) control points with position vectors \( p_0, p_1, \ldots, p_n \) in the plane, and let \( \tau := (t_0, t_1, \ldots, t_{n+k+1}) \) be a knot vector. Then the *(non-uniform) rational B-spline curve* of degree \( k \) (and order \( k + 1 \)) relative to \( \tau \) with control points \( p_0, p_1, \ldots, p_n \) is given by

\[
N(t) := \frac{\sum_{i=0}^{n} N_{i,k}(t) w_i p_i}{\sum_{i=0}^{n} N_{i,k}(t) w_i}
\]

for \( t \in [t_k, t_{n+1}] \),

where \( N_{i,k,\tau} \) is the \( i \)-th B-spline basis function of degree \( k \) relative to \( \tau \), with the weights \( w_i \in \mathbb{R}^+ \) for all \( i \in \{0, 1, \ldots, n\} \).

▶ If all \( w_i := 1 \) then we obtain the standard B-spline curve. (Recall the Partition of Unity, Cor. 125.)

▶ Both the numerator and the denominator are (piecewise) polynomials of degree \( k \). Hence, \( N \) is a piecewise rational curve of degree \( k \).

▶ In general, the weights \( w_i \) are required to be positive; a zero weight effectively turns off a control point, and can be used for so-called infinite control points [Piegl 1987].
Geometric Interpretation of NURBS

- We resort to homogeneous coordinates: For $w \neq 0$, \( \begin{pmatrix} u \\ v \\ w \end{pmatrix} \) are the homogeneous coordinates of \( \begin{pmatrix} x \\ y \end{pmatrix} \), and \( \begin{pmatrix} x \\ y \end{pmatrix} \) are the inhomogeneous coordinates of \( \begin{pmatrix} u \\ v \\ w \end{pmatrix} \) $\Longleftrightarrow [x = \frac{u}{w}$ and $y = \frac{v}{w}]$.

- For $p_i := \begin{pmatrix} x_i \\ y_i \end{pmatrix} \in \mathbb{R}^2$, let $p_i^w := \begin{pmatrix} w_i x_i \\ w_i y_i \\ w_i \end{pmatrix} \in \mathbb{R}^3$, for all $i \in \{0, 1, \ldots, n\}$.

- Now consider

$$N^w(t) := \sum_{i=0}^{n} N_{i,k}(t) p_i^w = \begin{pmatrix} \sum_{i=0}^{n} N_{i,k}(t)(w_i x_i) \\ \sum_{i=0}^{n} N_{i,k}(t)(w_i y_i) \\ \sum_{i=0}^{n} N_{i,k}(t) w_i \end{pmatrix}.$$ 

- Dividing the first two components of $N^w$ by its third component equals the (perspective) projection of $N^w$ to the plane $z = 1$.

- Hence, a NURBS curve in $\mathbb{R}^d$ is the projection of a B-spline curve in $\mathbb{R}^{d+1}$ and, thus, it inherits properties of B-spline curves.
Geometric Interpretation of NURBS

Projection onto $z = 1$

A NURBS curve in $\mathbb{R}^d$ is the projection of a B-spline curve in $\mathbb{R}^{d+1}$.

$\begin{align*}
(x_0, y_0, 1) &\to (w_0 x_0, w_0 y_0, w_0) \\
(x_1, y_1, 1) &\to (w_1 x_1, w_1 y_1, w_1) \\
(x_2, y_2, 1) &\to (w_2 x_2, w_2 y_2, w_2) \\
(x_3, y_3, 1) &\to (w_3 x_3, w_3 y_3, w_3)
\end{align*}$

▶ Note: A projection tends to increase rather than decrease continuity!
Rational (inhomogeneous) parametrization of the unit circle in the plane:

\[
x(t) := \frac{1 - t^2}{1 + t^2}
\]
\[
y(t) := \frac{2t}{1 + t^2}
\]

with \( t \in \mathbb{R} \).

Parametrization of the unit circle in the plane in homogeneous coordinates:

\[
u(t) := 1 - t^2
\]
\[
v(t) := 2t
\]
\[
w(t) := 1 + t^2
\]
NURBS Basis Functions

Definition 157 (NURBS basis function)

For $k \in \mathbb{N}_0$, weights $w_j > 0$ for all $j \in \{0, 1, \ldots, n\}$ and all (permissible) $i$, we define the $i$-th NURBS basis function of degree $k$ as

$$R_{i,k}(t) := \frac{N_{i,k}(t)w_i}{\sum_{j=0}^{n} N_{j,k}(t)w_j}.$$ 

▶ We can re-write the equation for $\mathcal{N}(t)$ as

$$\mathcal{N}(t) = \sum_{i=0}^{n} R_{i,k}(t)p_i \quad \text{for} \ t \in [t_k, t_{n+1}].$$

▶ Since NURBS basis functions in $\mathbb{R}^d$ are given by the projection of B-spline basis functions in $\mathbb{R}^{d+1}$, we may expect that the properties of B-spline basis functions carry over to NURBS basis functions.
Properties of NURBS Basis Functions

Lemma 158

For \( n \in \mathbb{N} \) and \( k \in \mathbb{N}_0 \) with \( k \leq n \), let \( \tau := (t_0, t_1, t_2, \ldots, t_{n+k+1}) \) be a knot vector. Then the following properties hold for all (permissible) values of \( i \in \mathbb{N}_0 \):

**Non-negativity:**
\[
R_{i,k}(t) \geq 0 \quad \text{for all real } t.
\]

**Local support:**
\[
R_{i,k}(t) = 0 \quad \text{if } t \notin [t_i, t_{i+k+1}].
\]

**Local influence:**
\[
R_{j,k} \text{ non-zero over } [t_i, t_{i+1}] \Rightarrow j \in \{i - k, i - k + 1, \ldots, i\}.
\]

**Partition of unity:**
\[
\sum_{j=0}^{n} R_{j,k}(t) = 1 \quad \text{for all } t \in [t_k, t_{n+1}].
\]

**Continuity:**
All NURBS basis functions of degree \( k \) are \( k - r \) times continuously differentiable at a knot of multiplicity \( r \), and \( k - 1 \) times continuously differentiable everywhere else.
Properties of NURBS Curves

Lemma 159

For \( n \in \mathbb{N} \) and \( k \in \mathbb{N}_0 \) with \( k \leq n \), consider a set of \( n + 1 \) control points with position vectors \( p_0, p_1, \ldots, p_n \) in the plane, and let \( \tau := (t_0, t_1, \ldots, t_{n+k+1}) \) be a knot vector. Then the following properties hold:

**Clamped interpolation:** If \( \tau \) is clamped then the NURBS curve \( \mathcal{N} \) starts in \( p_0 \) and ends in \( p_n \).

**Variation diminishing property:** If a straight line intersects the control polygon of \( \mathcal{N} \) \( m \) times then it intersects \( \mathcal{N} \) at most \( m \) times.

**Strong convex hull property:** For \( i \in \mathbb{N} \) with \( k \leq i \leq n \), we have

\[
\mathcal{N}|_{[t_i, t_{i+1}]} \subset CH(\{p_{i-k}, p_{i-k+1}, \ldots, p_{i-1}, p_i\}).
\]

**Local control:** The NURBS curve \( \mathcal{N} \) restricted to \( [t_i, t_{i+1}] \) depends only on the positions of \( p_{i-k}, p_{i-k+1}, \ldots, p_{i-1}, p_i \).

**Local modification scheme:** A modification of the position of \( p_i \) changes \( \mathcal{N} \) only in the parameter interval \([t_i, t_{i+k+1}]\), for \( i \in \{0, 1, \ldots, n\} \).
Properties of NURBS Curves

Lemma 160 (*Projective invariance*)

Any NURBS curve is projectively invariant, i.e., given any projective transformation \( \pi \), the image curve \( \pi(N) \) of a NURBS curve \( N \) with control points \( p_0, p_1, \ldots, p_n \) has the control points \( \pi(p_0), \pi(p_1), \ldots, \pi(p_n) \).

Lemma 161

For \( n \in \mathbb{N} \) and \( k \in \mathbb{N}_0 \) with \( k \leq n \), consider a set of \( n + 1 \) control points with position vectors \( p_0, p_1, \ldots, p_n \) in the plane, and let \( \tau := (t_0, t_1, \ldots, t_{n+k+1}) \) be a knot vector and \( w_0, w_1, \ldots, w_n \) be weights. Then the following properties hold for all \( i \in \{0, 1, \ldots, n\} \):

1. The weight \( w_i \) effects only the knot span \( [t_i, t_{i+k+1}] \).
2. If \( w_i \) decreases (relative to the other weights) then the NURBS curve is pushed away from \( p_i \).
3. If \( w_i = 0 \) then \( p_i \) does not contribute to the NURBS curve.
4. If \( w_i \) increases (relative to the other weights) then the NURBS curve is pulled towards \( p_i \).
Sample NURBS Curve

Clamped uniform rational B-spline of degree three for a control polygon with nine vertices:

\[
\left\{ \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 6 \\ 6 \end{pmatrix}, \begin{pmatrix} 8 \\ 8 \end{pmatrix}, \begin{pmatrix} 9 \\ 0 \end{pmatrix} \right\}
\]

Knot vector:

\[\tau := (0, 0, 0, 0, 1, 2, 3, 4, 5, 6, 6, 6, 6)\]

Weights:

\[(1, 1, 1, 1, 1, 1, 1)\]
Sample NURBS Curve

- Clamped uniform rational B-spline of degree three for a control polygon with nine vertices:

\[
\{ (-1, 0), (2, 3), (0, 6), (8, 8), (9, 0) \}
\]

Knot vector:

\[
\tau := (0, 0, 0, 0, 1, 2, 3, 4, 5, 6, 6, 6)
\]

Weights:

\[(1, 1, 1, 10, 1, 1, 1)\]
Sample NURBS Curve

- Clamped uniform rational B-spline of degree three for a control polygon with nine vertices:

\[
\{( -1, 0 ), ( 0, 0 ), ( 2, 2 ), ( 4, 4 ), ( 6, 6 ), ( 8, 8 ), ( 9, 9 ) \}
\]

Knot vector:

\[ \tau := ( 0, 0, 0, 0, 1, 2, 3, 4, 5, 6, 6, 6, 6 ) \]

Weights:

\[ (1, 1, 1, 0.1, 1, 1, 1) \]
Conics Modeled by NURBS

- NURBS can represent all conic curves — circle, ellipse, parabola, hyperbola — exactly.
- Conics are quadratic curves.
- Hence, consider three control points $p_0, p_1, p_2$ and the following quadratic NURBS curve

$$\mathcal{N}_2(t) := \frac{\sum_{i=0}^{2} N_{i,2}(t) w_i p_i}{\sum_{i=0}^{2} N_{i,2}(t) w_i} \quad \text{with } \tau := (0, 0, 0, 1, 1, 1),$$

i.e., a rational Bézier curve of degree two over $[0, 1]$.
- In expanded form we get

$$\mathcal{N}_2(t) = \frac{(1 - t^2)w_0 p_0 + 2t(1 - t)w_1 p_1 + t^2 w_2 p_2}{(1 - t^2)w_0 + 2t(1 - t)w_1 + t^2 w_2}.$$
- Can we come up with conditions for $w_0, w_1, w_2$ that allow to characterize the type of curve represented by $\mathcal{N}_2$?
Conics Modeled by NURBS

Lemma 162

The conic shape factor, \( \rho \), determines the type of conic represented by \( \mathcal{N}_2 \):

\[
\rho := \frac{w_1^2}{w_0 w_2} \begin{cases} < 1 & \ldots \text{\( \mathcal{N}_2 \) is an elliptic curve,} \\ = 1 & \ldots \text{\( \mathcal{N}_2 \) is a parabolic curve,} \\ > 1 & \ldots \text{\( \mathcal{N}_2 \) is a hyperbolic curve.} \end{cases}
\]

- Clamped uniform rational B-spline \( \mathcal{N}_2 \) of degree two with three control vertices
  \[
  \left\{ \left( \begin{array}{c} 1 \\ 0 \end{array} \right), \left( \begin{array}{c} 1 \\ 1 \end{array} \right), \left( \begin{array}{c} 0 \\ 1 \end{array} \right) \right\}
  \]
  and knots

  \[ \tau := (0, 0, 0, 1, 1, 1) \]

  and weights:

  \[ (1, 1/10, 1), \quad \text{hence} \quad \rho < 1. \]
Conics Modeled by NURBS

Lemma 162

The conic shape factor, $\rho$, determines the type of conic represented by $N_2$:

$$\rho := \frac{w_1^2}{w_0 w_2} \begin{cases} < 1 & \cdots \ N_2 \text{ is an elliptic curve,} \\ = 1 & \cdots \ N_2 \text{ is a parabolic curve,} \\ > 1 & \cdots \ N_2 \text{ is a hyperbolic curve.} \end{cases}$$

Clamped uniform rational B-spline $N_2$ of degree two with three control vertices

$$\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$$

and knots

$$\tau := (0, 0, 0, 1, 1, 1)$$

and weights:

$$(1, 1, 1), \quad \text{hence } \rho = 1.$$
Lemma 162

The *conic shape factor*, $\rho$, determines the type of conic represented by $\mathcal{N}_2$:

$$\rho := \frac{w_1^2}{w_0 w_2} \begin{cases} < 1 & \text{... } \mathcal{N}_2 \text{ is an elliptic curve}, \\ = 1 & \text{... } \mathcal{N}_2 \text{ is a parabolic curve}, \\ > 1 & \text{... } \mathcal{N}_2 \text{ is a hyperbolic curve}. \end{cases}$$

- Clamped uniform rational B-spline $\mathcal{N}_2$ of degree two with three control vertices
  \[ \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\} \]
  and knots
  \[ \tau := (0, 0, 0, 1, 1, 1) \]
  and weights:
  \[ (1, 5, 1), \text{ hence } \rho > 1. \]
Lemma 163

The quadratic NURBS curve $N_2$ represents a circular arc if the control points $p_0, p_1, p_2$ form an isosceles triangle, and if the weights are set as follows:

$$w_0 := 1, \quad w_1 := \frac{||p_0 - p_2||}{2 \cdot ||p_0 - p_1||}, \quad w_2 := 1$$

- The weight $w_1$ is related to the central angle $\varphi$ subtended by the arc: $w_1 = \cos(\varphi/2)$.
- We can join four quarter-circle NURBS to form a full circle.
- In this case, the isosceles triangles defining the quarter circles need to add up to a square.
It is also possible to construct a circle by a single NURBS curve.

\[
\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}
\]

Knots:
\[
(0, 0, 0, \frac{\pi}{2}, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, \frac{3\pi}{2}, 2\pi, 2\pi, 2\pi)
\]

Weights:
\[
(1, \frac{1}{\sqrt{2}}, 1, \frac{1}{\sqrt{2}}, 1, \frac{1}{\sqrt{2}}, 1, \frac{1}{\sqrt{2}}, 1)
\]

Note: The positioning of the control points ensures that the first derivative is continuous, despite of double knots.

Note: \(N'(t) \neq (\sin t, \cos t)\) for \(t \neq \frac{m \cdot \pi}{4}\).
Applying an affine transformation to the control points yields an ellipse.

\[
\left\{ \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \end{pmatrix} \right\}
\]

Knots: (0, 0, 0, 1, 1, 2, 2, 3, 3, 4, 4, 4)  Weights: (1, $\frac{1}{\sqrt{2}}$, 1, $\frac{1}{\sqrt{2}}$, 1, $\frac{1}{\sqrt{2}}$, 1, $\frac{1}{\sqrt{2}}$, 1)
Sample NURBS Surface
Approximation and Interpolation

Distance Measures

Interpolation and Approximation of Point Data

Bernstein Approximation of Functions
Hausdorff Distance

- Let $A, B$ be two subsets of a metric space $X$ and let $d(p, q)$ denote the distance between two elements $p, q \in X$. E.g., take $\mathbb{R}^n$ and the (standard) Euclidean distance.
- How can we measure how similar $A$ and $B$ are?
- This is a frequently asked question in image processing, solid modeling, computer graphics and computational geometry.
- Note that the classical minimin function

$$D(A, B) := \inf_{a \in A} \left( \inf_{b \in B} d(a, b) \right)$$

is a very poor measure of similarity between $A$ and $B$: One can easily get $D(A, B) = 0$ although $A$ and $B$ need not be similar at all, according to any natural human interpretation of similarity.
- So, can we do any better?
Hausdorff Distance

Definition 164 (*Hausdorff distance*)

Let $A, B$ be two non-empty subsets of a metric space $X$ and let $d$ be any metric on $X$. The *directed Hausdorff distance*, $h(A, B)$, from $A$ to $B$ is a maximin function, defined as

$$h(A, B) := \sup_{a \in A} \left( \inf_{b \in B} d(a, b) \right).$$

The *(symmetric) Hausdorff distance*, $H(A, B)$, between $A$ and $B$ is defined as

$$H(A, B) := \max \{ h(A, B), h(B, A) \}.$$  

- Introduced by Felix Hausdorff in 1914.
- If both $A$ and $B$ are bounded then $H(A, B)$ is guaranteed to be finite.
- For compact sets we can replace $\inf$ by $\min$ and $\sup$ by $\max$.
- The function $H$ defines a metric on the set of all non-empty compact subsets of a metric space $X$.
- For sets of $n$ points in $\mathbb{R}^2$, the Hausdorff distance can be computed in time $O(n \log n)$, using a Voronoi-based approach $\rightarrow$ computational geometry.
- A common variation is the *Hausdorff distance under translation*. 
Fréchet Distance

- The Hausdorff distance does not capture any form of orientation or continuity, as we might be interested in when matching curves or surfaces.

**Definition 165 (Fréchet distance)**

Consider a closed interval $I \subset \mathbb{R}$ and two curves $\beta, \gamma : I \rightarrow \mathbb{R}^n$. The Fréchet distance between $\beta(I)$ and $\gamma(I)$ is defined as

$$\text{Fr}(\beta, \gamma) := \inf_{\sigma, \tau} \max_{t \in I} \| \beta(\sigma(t)) - \gamma(\tau(t)) \|,$$

where $\sigma, \tau : I \rightarrow I$ range over all continuous and monotonously increasing functions that map $I$ to $I$ such that $\sigma(I) = I$ and $\tau(I) = I$.

- Popular interpretation [Alt&Godau 1995]: Suppose that a person is walking a dog. Assume the person is walking on one curve and the dog on another curve. Both can adjust their speeds but are not allowed to move backwards.

- We can think of the parameter $t$ as time: Then $\beta(\sigma(t))$ is the position of the person and $\gamma(\tau(t))$ is the position of the dog at time $t$. The length of the leash between them at time $t$ is the distance between $\beta(\sigma(t))$ and $\gamma(\tau(t))$.

- Then the Fréchet distance of the two curves is the minimum leash length necessary to keep the person and the dog connected.
Fréchet Distance

- Note that we do not demand strict monotonicity for either $\sigma$ or $\tau$.
- While efficient algorithms are known for computing the Fréchet distance of polygonal curves, the same problem is $\mathcal{NP}$-hard for triangulated surfaces.
- However, a variant, the so-called weak Fréchet distance, can be computed in polynomial time [Alt&Buchin 2010].
- The Fréchet distance between two polygonal curves may be arbitrarily larger than the Hausdorff distance between them.
Interpolation Versus Approximation

- For \( m \in \mathbb{N}_0 \), we are given \( m + 1 \) points \( q_0, q_1, \ldots, q_m \in \mathbb{R}^n \), possibly with matching parameter values \( u_0 < u_1 < \ldots < u_m \).

- For an interpolation of \( q_0, q_1, \ldots, q_m \) we seek a curve \( C \) such that either
  - \( C(x_i) = q_i \) for arbitrary \( x_i \in \mathbb{R} \), for all \( i \in \{0, 1, \ldots, m\} \), or
  - \( C(u_i) = q_i \) for all \( i \in \{0, 1, \ldots, m\} \).

- For an approximation of \( q_0, q_1, \ldots, q_m \) we seek a curve \( C \) such that the distance between \( C \) and \( q_0, q_1, \ldots, q_m \) is smaller than a user-specified threshold relative to some distance measure.

- Similarly for approximation/interpolation by a surface rather than a curve.
Humorous View of Approximation

[Image credit: https://xkcd.com]
Lagrange Interpolation

Definition 166 (Lagrange polynomial)

For $m \in \mathbb{N}$, consider $m + 1$ parameter values $u_0 < u_1 < \ldots < u_m$ and let $i \in \{0, 1, \ldots, m\}$. Then the $i$-th Lagrange polynomial of degree $m$ is defined as

$$L_{i,m}(u) := \prod_{j=0, i \neq j}^{m} \frac{u - u_j}{u_i - u_j}.$$ 

Definition 167 (Lagrange interpolation)

For $m \in \mathbb{N}$, consider $m + 1$ parameter values $u_0 < u_1 < \ldots < u_m$ and $m + 1$ data points $q_0, q_1, \ldots, q_m$. Then the Lagrange interpolation of $q_0, q_1, \ldots, q_m$ is given by

$$\mathcal{L}(u) := \sum_{i=0}^{m} L_{i,m}(u) q_i.$$
Lagrange Interpolation

Lemma 168

For $m \in \mathbb{N}$, let $\mathcal{L}$ be defined for $m + 1$ parameter values $u_0 < u_1 < \ldots < u_m$ and $m + 1$ data points $q_0, q_1, \ldots, q_m$, as given in Def. 167. Then $\mathcal{L}(u_k) = q_k$ for all $k \in \{0, 1, \ldots, m\}$.

Proof: For all $k \in \{0, 1, \ldots, m\}$, we have

$$L_{i,m}(u_k) = \prod_{j=0, i \neq j}^{m} \frac{u_k - u_j}{u_i - u_j} = \delta_{ik} = \begin{cases} 0 & \text{if } i \neq k, \\ 1 & \text{if } i = k. \end{cases}$$

Hence,

$$\mathcal{L}(u_k) = \sum_{i=0}^{m} L_{i,m}(u_k)q_i = q_k.$$

Corollary 169

For $m \in \mathbb{N}$, the Lagrange polynomials $L_{0,m}, L_{1,m}, \ldots, L_{m,m}$ form a basis of the vector space of all polynomials of degree at most $m$.

Sketch of Proof: Recall that exactly one polynomial of degree $m$ interpolates $m + 1$ data points.
**Newton Interpolation**

**Definition 170 (Newton polynomial)**

For \( m \in \mathbb{N} \), consider \( m + 1 \) parameter values \( u_0 < u_1 < \ldots < u_m \) and let \( i \in \{0, 1, \ldots, m\} \). Then the \( i \)-th Newton polynomial is defined as

\[
I_i(u) := \prod_{j=0}^{i-1} (u - u_j)
\]

with, by convention, \( I_0(u) := 1 \).

**Definition 171 (Newton interpolation)**

For \( m \in \mathbb{N} \), consider \( m + 1 \) parameter values \( u_0 < u_1 < \ldots < u_m \) and \( m + 1 \) data points \( q_0, q_1, \ldots, q_m \). Then the Newton interpolation of \( q_0, q_1, \ldots, q_m \) is given by

\[
I(u) := \sum_{i=0}^{m} I_i(u)p_i,
\]

with

\[
p_i := \begin{cases} 
q_i & \text{for } i = 0, \\
\frac{q_i - \sum_{j=0}^{i-1} l_j(u_i)p_j}{l_i(u_i)} & \text{for } i > 0.
\end{cases}
\]
Newton Interpolation

Lemma 172

For $m \in \mathbb{N}$, let $I$ be defined for $m + 1$ parameter values $u_0 < u_1 < \ldots < u_m$ and $m + 1$ data points $q_0, q_1, \ldots, q_m$, as given in Def. 171. Then $I(u_k) = q_k$ for all $k \in \{0, 1, \ldots, m\}$.

Proof: For all $k \in \{0, 1, \ldots, m\}$, we have for all $i > 1$

$$l_i(u_k) = \prod_{j=0}^{i-1} (u_k - u_j) \begin{cases} = 0 & \text{if } i \geq k + 1, \\ \neq 0 & \text{if } i \leq k. \end{cases}$$

We have

$$I(u_0) = 1 \cdot p_0 = q_0,$$

and for each $1 \leq k \leq m$

$$I(u_k) = \sum_{i=0}^{m} l_i(u_k) p_i = \sum_{i=0}^{k} l_i(u_k) p_i = \sum_{i=0}^{k-1} l_i(u_k) p_i + l_k(u_k) p_k$$

$$= \sum_{i=0}^{k-1} l_i(u_k) p_i + l_k(u_k) \cdot \frac{q_k - \sum_{j=0}^{k-1} l_j(u_k) p_j}{l_k(u_k)} = q_k.$$
Sampling of a function $f$ and subsequent Lagrange interpolation may yield an extremely poor approximation of $f$ even if $f$ is continuously differentiable.

C. Runge: Consider $f(x) := \frac{1}{1 + x^2}$ and $n + 1$ uniform samples within $[-5, 5]$, with $n := 20$.

Similar problems occur for Newton interpolation.
B-Spline Interpolation

Let $k \in \mathbb{N}_0$ and suppose that we are looking for $n + 1$ control points $p_0, p_1, \ldots, p_n$ and a knot vector $\tau := (t_0, t_1, \ldots, t_{n+k+1})$ such that the B-spline curve $B$ of degree $k$ defined by $p_0, p_1, \ldots, p_n$ and $\tau$ interpolates $q_0, q_1, \ldots, q_m$, with $B(u_i) = q_i$ for all $i \in \{0, 1, \ldots, m\}$ and some given $u_0 < u_1 < \ldots < u_m$.

If $n = m$, then we get the following system of equations:

\[
\begin{pmatrix}
N_{0,k}(u_0) & \cdots & N_{n,k}(u_0) \\
\vdots & \ddots & \vdots \\
N_{0,k}(u_n) & \cdots & N_{n,k}(u_n)
\end{pmatrix}
\begin{pmatrix}
p_0 \\
p_1 \\
p_n
\end{pmatrix}
=:\begin{pmatrix}
q_0 \\
q_1 \\
q_n
\end{pmatrix}
\]

Hence, the interpolation problem can be solved if the (quadratic) collocation matrix $\mathbf{N}$ is invertible.

Lemma 173 (Schönberg-Whitney)

The collocation matrix $\mathbf{N}$ is invertible if and only if if all its diagonal elements are non-zero, i.e., if and only if $t_i \leq u_i < t_{i+k+1}$, for all $i \in \{0, 1, \ldots, n\}$.

Lem. 117: The matrix $\mathbf{N}$ is a sparse band matrix without negative elements.

Fast and numerically reliable algorithms exist for computing the inverse of $\mathbf{N}$. 
Most applications do not require specific parameter values \(u_i\).

In such a case, one can fix the knots \(t_i\), and choose \(u_i\) as follows ("Greville-abscissae"):

\[
u_i := \frac{1}{k} \sum_{j=1}^{k} t_{i+j} \quad \text{for all } i \in \{0, 1, \ldots, n\}.
\]

Note that \(t_i\) and \(t_{i+k+1}\) do not enter the definition of \(u_i\).

Of course,

\[
t_i \leq t_{i+1} \leq \frac{1}{k} (t_{i+1} + \cdots + t_{i+k}) \leq t_{i+k} \leq t_{i+k+1},
\]

thus meeting the Schönberg-Whitney condition of Lem. 173. Equality would only occur if an inner knot has multiplicity \(k + 1\). (But then the B-spline would be discontinuous!)
Effects of Parameters and Knots

Since a B-spline has continuous speed and acceleration (for \( k \geq 3 \)), it is obvious that the parameter values \( u_i \) should bear a meaningful relation to the distances between the data points. Otherwise, overshooting is bound to occur!

Consider

\[ u_0 := 0 \quad \text{and} \quad u_{i+1} := u_i + \Delta_i \quad \text{for all} \quad i \in \{1, \ldots, m - 1\}, \]

with

\[ \Delta_i := \|q_i - q_{i-1}\|^p \quad \text{for some} \quad p \in [0, 1] \quad \text{and all} \quad i \in \{1, \ldots, m - 1\}. \]

These parameter values are known as **uniform** if \( p = 0 \), **centripetal** if \( p = \frac{1}{2} \), and **chordal** if \( p = 1 \).

Suitable knots that meet the Schönberg-Whitney conditions (Lem. 173) are defined as follows:

\[ t_i := \frac{1}{k} \left( u_{i-k} + u_{i-k+1} + \ldots + u_{i-1} \right) \]
B-Spline Approximation

- If $m > n$, i.e., if there are more data points than control points, then the linear system $Np = q$ is over-determined and a solution need not exist.

- One popular option is a least-squares fit, which is achieved if

  $$N^T N p = N^T q.$$  

- Hence, if $N^T N$ is invertible then we get

  $$p = (N^T N)^{-1} N^T q.$$  

- An extension of the Schönberg-Whitney Lem. 173 tells us that the matrix $N^T N$ is invertible exactly if the Schönberg-Whitney conditions are met:

**Lemma 174**

The matrix $N^T N$ is invertible if and only if $t_i \leq u_i < t_{i+k+1}$, for all $i \in \{0, 1, \ldots, n\}$.  

Bernstein Polynomials

Definition 175 (*Bernstein polynomial*)

For \( n \in \mathbb{N}_0 \), a *Bernstein polynomial* of degree \( n \) is a linear combination of Bernstein basis polynomials of degree \( n \):

\[
B_n(x) := \sum_{i=0}^{n} \mu_i B_{i,n}(x), \quad \text{with } \mu_0, \mu_1, \ldots, \mu_n \in \mathbb{R}.
\]

- Hence, every polynomial (in power basis) can be seen as a Bernstein polynomial, albeit with unknown scalars for the linear combination.
- Can we select \( \mu_i \) such that a decent approximation of a user-specified function is achieved?
**Bernstein Approximation**

**Definition 176 (Bernstein approximation)**

Consider a continuous function $f : [0, 1] \rightarrow \mathbb{R}$. For $n \in \mathbb{N}$, the Bernstein approximation with degree $n$ of $f$ is defined as

$$B_{n,f}(x) := \sum_{i=0}^{n} f\left(\frac{i}{n}\right) B_{i,n}(x).$$

▶ Hence, a Bernstein approximation is given by a Bernstein polynomial, with weights $\mu_i := f\left(\frac{i}{n}\right)$.

**Theorem 177 (Weierstrass 1885, Bernstein 1911)**

The Bernstein approximation $B_{n,f}$ converges uniformly to the continuous function $f$ on the interval $[0, 1]$. That is, given a tolerance $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$|f(x) - B_{n,f}(x)| \leq \varepsilon \quad \text{for all } x \in [0, 1] \text{ and all } n \geq n_0.$$

▶ Since $x := \frac{t-a}{b-a}$ maps $t \in [a, b]$ to $x \in [0, 1]$, this approximation theorem extends to continuous functions $f : [a, b] \rightarrow \mathbb{R}$. 
Sample Bernstein Approximation

- Sample Bernstein approximation of a continuous function:

$f: [0, 1] \rightarrow \mathbb{R}$

\[ f(x) := \frac{1}{1 + (10x - 5)^2} \]
Sample Bernstein Approximation

Sample Bernstein approximation of a continuous function:

\[ f : [0, 1] \rightarrow \mathbb{R} \]

\[ f(x) := \frac{1}{1 + (10x - 5)^2} \]

Degree 128
Sample Bernstein Approximation

Sample Bernstein approximation of a continuous function:

\[ f : [0, 1] \to \mathbb{R} \]

\[ f(x) := \sin (\pi x) + \frac{1}{5} \sin \left(6\pi x + \pi x^2\right) \]
Sample Bernstein Approximation

Sample Bernstein approximation of a continuous function:

\[ f: [0, 1] \to \mathbb{R} \]

\[ f(x) := \sin (\pi x) + \frac{1}{5} \sin \left( 6\pi x + \pi x^2 \right) \]
The End!

I hope that you enjoyed this course, and I wish you all the best for your future studies.