Assignment 1 Familiarize yourself with the $O$-notation. (You should be able to explain the terms $O(f), \Omega(f), \Theta(f), o(f)$ for functions $f: \mathbb{N} \to \mathbb{R}^+$. (See, e.g., my slides on Discrete Mathematics.)

Assignment 2 Consider functions $f, g: \mathbb{N} \to \mathbb{R}$, with $f(n) := n \log n - 1$ and $g(n) := 10n + 5\sqrt{n} + 3$, and prove the following claims:

1. $g \in O(f)$.
2. $g \notin \Theta(f)$.
3. $g \in o(f)$.

Assignment 3 Consider $n + 1$ distinct real numbers $a_0, a_1, \ldots, a_n$. We know that $0 = a_0 < a_i < a_n = 1$ for all $0 < i < n$. (However, no sorting of these numbers is known!) These numbers split the interval $[0, 1]$ into $n + 1$ (topologically open) sub-intervals. We color all odd-numbered intervals black and all even-numbered intervals white. Then we ask whether all $n$ numbers of a set $Q := \{q_1, \ldots, q_n\}$ of query numbers lie in black intervals. (No sorted order of $Q$ is known, but you may assume that no $q_j$ is identical to an $a_i$.) What is a (non-trivial) lower bound on the complexity of this query in the ADT model of computation in the worst case? Can you come up with an algorithm to solve this query that has a matching worst-case complexity?

Assignment 4 Consider $n + 1$ distinct real numbers $a_0, a_1, \ldots, a_n$. We know that $0 = a_0 < a_i < a_n = 1$ for all $0 < i < n$. (However, no sorting of these numbers is known!) These numbers split the interval $[0, 1]$ into $n + 1$ (topologically open) sub-intervals. Now we ask whether all $n$ numbers of a set $Q := \{q_1, \ldots, q_n\}$ of query numbers lie in the same sub-interval. (No sorted order of $Q$ is known, but you may assume that no $q_j$ is identical to an $a_i$.) What is a (non-trivial) lower bound on the complexity of this query in the worst case? Can you come up with an algorithm to solve this query that has a matching worst-case complexity?
Assignment 5 Let \( f_1, f_2, g_1, g_2 : \mathbb{N} \to \mathbb{R}^+ \). Prove or disprove the following claims.

1. \( o(f_1) \subset O(f_1) \) and \( o(f_1) \neq O(f_1) \),
2. \( [g_1 \in \Omega(f_1) \land g_2 \in \Omega(f_2)] \implies (g_1 \cdot g_2) \in \Omega(f_1 \cdot f_2) \).

Assignment 6 Let \( L \) denote a set of \( n \) lines in the plane and let \( A(L) \) denote their arrangement. (That is, \( A(L) \) is the planar subdivision introduced by intersecting all lines of \( L \) and splitting them at their points of intersection.) Describe an \( O(n \log n) \) algorithm to compute an axis-parallel rectangle that contains all the vertices of \( A(L) \).

Assignment 7 Describe an \( O(n) \) algorithm for determining a point that lies inside of the kernel of an \( n \)-vertex star-shaped polygon. (Simply stating that this can be done by means of linear programming is not good enough — you don’t even need to resort to an LP-based approach!)

Assignment 8 Let \( S \) be a set of \( n \) disjoint line segments in the plane and let \( p \) be a point which does not lie on any segment in \( S \). The goal is to determine in \( O(n \log n) \) all the line segments that are visible from \( p \). (A segment \( \ell \) in \( S \) is visible from \( p \) if and only if there exists a point \( q \) on \( \ell \) such that the segment \( pq \) intersects only segment \( \ell \).)
PS Computational Geometry

Homework Assignment Sheet III (Due 14-Nov-2014)

**Assignment 9** Explain a simple and reliable algorithm for testing in $O(n)$ time whether a point is inside of an $n$-vertex simple polygon. Please note that your algorithm has to be able to cope with degeneracies.

**Assignment 10** Let $S$ be a set of $n$ axis-aligned squares, and let $U := \bigcup_{s \in S} s$. Prove that the combinatorial complexity of the boundary $\partial U$ of $U$ is $O(n)$. That is, prove that $\partial U$ has $O(n)$ many vertices and edges. Note that the squares need not be of equal size! Does this bound also carry over to $n$ axis-aligned rectangles?

**Assignment 11** Given are $n$ numbers $x_1, \ldots, x_n \in \mathbb{R}$. Prove that one can choose a set $S := \{p_1, \ldots, p_n\}$ of $n$ distinct points in $\mathbb{R}^2$ in $O(n)$ time such that

1. the set $S$ is in convex position, i.e., all points of $S$ form vertices of $CH(S)$,
2. for all $1 \leq i \leq n$, the $x$-coordinate of point $p_i$ equals $x_i$.

**Assignment 12** Prove directly that the nucleus of a simple polygon always is path-connected, without resorting to arguments that make use of the fact that the nucleus is given by the intersection of half-planes.

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Homework Assignment Sheet IV (Due 21-Nov-2014)

Assignment 13 Given a set of \( n \) points in the plane, devise an algorithm that checks whether there exists a subset of three points which are collinear. Your algorithm should run in time \( O(n^2 \log n) \), or better.

Assignment 14 Consider a set \( S \) of \( n \) points in \( \mathbb{R}^3 \) and assume that their convex hull \( CH(S) \) is known. For a plane \( \varepsilon \), let \( S' \) be the set of points obtained by projecting all points of \( S \) (by means of an orthogonal projection) onto \( \varepsilon \). Does the knowledge of \( CH(S) \) help to compute \( CH(S') \) in \( o(n \log n) \) time? (You are welcome to assume that the points of \( S \) are in general position. Furthermore, you may assume that \( CH(S) \), which is a polytope, is stored in a “reasonable” manner that supports standard navigational operations on its boundary; e.g., you may assume that \( CH(S) \) is stored as a DCEL or in a winged-edge data structure.)

Assignment 15 Given are \( n \) numbers \( x_1, \ldots, x_n \in \mathbb{R} \). Prove constructively that one can choose a set \( S := \{p_1, \ldots, p_n\} \) of \( n \) distinct points in \( \mathbb{R}^2 \) in \( O(n) \) time such that for all \( 1 \leq i \leq n \), the \( x \)-coordinate of point \( p_i \) equals \( x_i \), and such that the convex hull \( CH(S) \) and a triangulation of \( S \) can be obtained in \( O(n) \) time, too.

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Assignment 16 In the PS on 07-Nov-2014 we discussed whether a point within the nucleus of a star-shaped polygon $P$ can be found in an iterative manner, via repeated orthogonal projection: Start with a candidate point chosen randomly within (the interior of) $P$ and check whether it is within all the half-planes that define the constraints for the nucleus. If it is outside of one or more half-planes, then select such a half-plane (by some heuristic) and project the point orthogonally onto the line that defines the half-plane, thus obtaining a new candidate point. Repeat until a point within the nucleus of $P$ is obtained. Is this procedure guaranteed to terminate in a finite number of projection steps for every input polygon $P$ and every start point within $P$? (You are free to choose an appropriate strategy for selecting a half-plane, if such a good strategy exists. Also, there is no need to try to come up with an $O$-term that models the number of steps.)

Assignment 17 Prove that the computation of the convex hull of an $n$-vertex star-shaped polytope in $\mathbb{R}^3$ vertices requires $\Omega(n \log n)$ time in the worst case.

Assignment 18 Suppose that three (straight-line) rays start at the same point $v$ in the plane. Hence, these rays partition the plane into three wedges $W_1, W_2, W_3$. Can you compute a set $S := \{p_1, p_2, p_3\}$ of three points $p_1, p_2, p_3$ such that $W_i = VR(p_i, S)$ for $1 \leq i \leq 3$? Is this always possible? If a solution $S$ exists, is it unique?
Assignment 19 Consider $\mathcal{M} := \{0, 1, 2, \ldots, n - 1\} \times \{0, 1, 2, \ldots, n - 1\}$ and suppose that $k$ of these $n^2$ many points are colored black while all other $n^2 - k$ points are colored white, for some $0 \leq k \leq n^2$. In this discrete world, distance is measured in the $L_\infty$ metric. We would like to identify for every black point $b$ all white points whose distance to $b$ is less than the distance to any other black point, i.e., the Voronoi region of $b$. Is it possible to determine all Voronoi regions of all black points in $o(n^2 \log n)$ time?

Assignment 20 It is straightforward to generalize the definition of a Delaunay triangulation of points from $\mathbb{R}^2$ to $\mathbb{R}^3$, with Delaunay triangles becoming Delaunay tetrahedra. (That is, we get a Delaunay edge in $\mathbb{R}^3$ as the dual of a Voronoi face.) However, not all properties carry over from $\mathbb{R}^2$ to $\mathbb{R}^3$. Prove that the Delaunay triangulation of a set of $n$ points in $\mathbb{R}^3$ may contain $\Theta(n^2)$ many Delaunay tetrahedra.

Assignment 21 Let $\mathcal{P}$ be a simple polygon. Use a parity argument to prove directly — without resorting to the Jordan Curve Theorem — that $\mathbb{R}^2 \setminus \mathcal{P}$ has at least two connected components.
Assignment 22 Recall that an approximate TSP tour for a set $S$ of $n$ points in the Euclidean plane can be obtained by means of the doubling-the-EMST heuristic as follows: compute $\text{EMST}(S)$, pick one point of $S$ (and, thus, one node of $\text{EMST}(S)$) as a root, and apply a standard tree traversal – e.g., a pre-order traversal – to $\text{EMST}(S)$, where nodes already visited are bypassed. Theory tells us that the approximation factor achieved by this heuristic is no worse than 2. Show that there exist configurations of $n$ points for which the approximation factor will indeed approach 2 as $n$ grows, no matter which node of $\text{EMST}(S)$ is picked as the root. Similarly, show that there exist configurations of $n$ points for which the approximation factor achieved by Christofides’ heuristic approaches $\frac{3}{2}$. Again, the approximation factor achieved has to be roughly $\frac{3}{2}$ no matter at which node the Eulerian tour starts. (No formal proofs are needed, but your point configurations should be simple enough to admit reasonably intuitive and clean arguments. You are welcome to restrict your attention to values of $n$ that are all even or all odd, if this helps. However, it has to be clear how your point configurations would be specified for arbitrarily large values of $n$.)

Assignment 23 We define the lune over a straight-line segment $pq$ in $\mathbb{R}^2$ as the set of points $\{r \in \mathbb{R}^2 : d(p,q) \geq \max\{d(p,r),d(q,r)\}\}$. Prove that the lune over an edge of an EMST never contains any other input point in its interior.

Assignment 24 Consider two distinct points $p, q \in \mathbb{R}^2$. We define the distance of $u \in \mathbb{R}^2$ from $p$ as $w_p d(p,u)$, where $d(\cdot, \cdot)$ denotes the standard Euclidean distance and $w_p \in \mathbb{R}^+$. Similarly, $w_q d(q,u)$ gives the distance between $u$ and $q$, for $w_q \in \mathbb{R}^+$ with $w_p \neq w_q$. Prove that the bisector of $p$ and $q$ is a full circle. (Hint: An analytic proof works.)

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Assignment 25 Consider a set $S$ of $n$ points $p_1, p_2, \ldots, p_n$ in $\mathbb{R}^2$. Let $1 \leq i < j \leq n$ and assume that $e(i, j)$ is an edge of $VD(S)$ defined by $p_i$ and $p_j$. Prove:

1. Except for its two endpoints — i.e., Voronoi nodes — the edge $e(i, j)$ is contained entirely in the interior of the second-order Voronoi region $\mathcal{VR}(p_i, p_j, S)$ defined by $p_i$ and $p_j$.

2. If an edge of $VD(S)$ intersects the second-order Voronoi region $\mathcal{VR}(p_i, p_j, S)$ then this is $e(i, j)$.

Assignment 26 Consider a set $S$ of $n$ points $p_1, p_2, \ldots, p_n$ in $\mathbb{R}^2$. Use the properties established in Ass. 25 to compute the second-order Voronoi diagram of $S$ in $O(n \log n)$ time, by resorting to standard first-order Voronoi computations.

Assignment 27 Consider two distinct points $A, B \in \mathbb{R}^2$, and $w_A, w_B \in \mathbb{R}^+$ with $w_A \neq w_B$. For $t \in \mathbb{R}^+$ we consider the disk $D_A(t)$ centered at $A$ with radius $t \cdot w_A$. Similarly for $D_B(t)$ relative to $B$ and $w_B$. Let $O(t) := CH(D_A(t) \cup D_B(t))$. It is easy to see that $\partial O(t)$ consists of two straight-line segments and two circular arcs for all those $t \in \mathbb{R}^+$ for which neither $D_A(t) \subset D_B(t)$ nor $D_B(t) \subset D_A(t)$. Consider the common endpoint $p(t)$ of a circular arc and a straight-line segment of $\partial O(t)$, and prove that $p(t)$ lies on a half-circle for all $t \in \mathbb{R}^+$ for which neither $D_A(t) \subset D_B(t)$ nor $D_B(t) \subset D_A(t)$. (Hint: A geometric proof works.)

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Assignment 28 Let $\mathcal{P}$ be a simple polygon. Let $\delta$ be the minimum distance from any vertex $v$ of $\mathcal{P}$ to any edge of $\mathcal{P}$ that is not incident to $v$, and let $0 < \varepsilon < \delta/2$. Prove that the offset $\mathcal{P}_\varepsilon$ of $\mathcal{P}$ for offset distance $\varepsilon$ consists of precisely two closed curves.

Assignment 29 Consider three distinct points $p_1, p_2, p_3 \in \mathbb{R}^2$ which are not collinear, and three direction vectors $v_1, v_2, v_3 \in \mathbb{R}^2$. (These vectors have non-zero length but they need not be unit vectors!) For $t \in \mathbb{R}^+_0$, these three points and vectors define three points $p_i(t) := p_i + t \cdot v_i$ that move away from $p_i$, for $i \in \{1, 2, 3\}$. Prove that $p_1(t), p_2(t), p_3(t)$ can be collinear at most twice during their entire movement.

Assignment 30 Prove that $(A \oplus B) \ominus B$ and $(A \ominus B) \oplus B$ need not equal $A$ for all sets $A, B$, where $\oplus$ and $\ominus$ denote the Minkowski sum and difference, respectively.
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Homework Assignment Sheet X (Due 16-Jan-2015)

Assignment 31 Let $P$ be a simple polygon. Study offsets of $P$ and use them to prove directly — without resorting to the Jordan Curve Theorem — that $\mathbb{R}^2 \setminus P$ has at most two connected components.

Assignment 32 Prove that the Delaunay triangulation of a set $S$ of points need not also form a minimum-weight triangulation of $S$.

Assignment 33 Let $S$ be a set of points in $\mathbb{R}^2$. Prove explicitly that the minimum internal angle of the triangles of $\mathcal{DT}(S)$ is maximum over all triangulations of $S$.

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Assignment 34 Does the Delaunay triangulation of a set $S$ of points in $\mathbb{R}^2$ minimize the maximum internal angle over all triangulations of $S$?

Assignment 35 Let $P$ be a polygon whose vertices all are co-circular, and let $T$ be a triangulation of $P$. Suppose that a triangle $t$ of $T$ has the smallest interior angle among all triangles of $T$. Prove that $t$ contains at least one edge of $P$ as edge.

Assignment 36 Suppose that a set $S$ of points in the plane contains a subset $S'$ such that all points of $S'$ are co-circular and such that no other point of $S$ lies within $CH(S')$. Obviously, $DT(S)$ is not uniquely defined in such a case. Prove that $S'$ can be triangulated arbitrarily without violating the property of Delaunay triangulations that the minimum interior angle of the triangulation is maximized over all triangulations.
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Homework Assignment Sheet XII (Due 30-Jan-2015)

Assignments for this due date may still be modified at the instructor’s discretion!