Composition of Rules of Frege-Hilbert Calculi

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1. Introduction

Inference mechanisms based on backward reasoning in cut-free sequent calculi have been widely adopted for systems of automated as well as interactive deduction such as the connection method [2] or tableau systems [1]. Since application of the cut rule can dramatically shorten proofs as Statman [5] and Orevkov have shown, it seems desirable to incorporate a cut rule in such systems. For automated reasoning this, however, poses the problem that there is an infinite number of possible backward applications of the cut rule at each step of backward reasoning, yielding the backward proof search intractable. A remedy is to compute derivation schemes instead of derivations [3, 4]. For example, instead of proving $\neg\neg\neg P \vdash \neg P$ where P is a proposition symbol, we prove a general rule '$\frac{\neg\neg A}{A}$' where 'A' is a metavariable standing for an arbitrary formula. A proof is constructed by repeatedly building the composition of two such rules using unification on metavariables. In this paper the algebra of rules is studied for Frege-Hilbert calculi which have a simpler mathematical structure than sequent calculi. The results carry over to sequent calculi which are better suited for automation as well as interactive systems.

Let $\mathbb{S}$ be the set of *sentences*, i.e. of closed formulas. A *rule* R defines a subset $\mathbb{I}_R$ of $\mathbb{S}^n \times \mathbb{S}$ where $n \geq 0$ is the arity of R. The elements of $\mathbb{I}_R$ are called *instances* of R. The *k-th composition* of two rules R and $\hat{R}$ is the rule whose instances are those pairs $\langle\langle B_1, \ldots, B_{k-1}, A_1, \ldots, A_m, B_{k+1}, \ldots, B_n\rangle, C\rangle$ for which there is a B_k such that $\langle\langle A_1, \ldots, A_m\rangle, B_k\rangle \in \mathbb{I}_R$ and $\langle\langle B_1, \ldots, B_n\rangle, C\rangle \in \mathbb{I}_{\hat{R}}$. A *Frege-Hilbert calculus* is given by a finite set of rules. A rule is *composable in* a calculus $\mathcal{C}$, or $\mathcal{C}$-composable, if it is a composition of rules of $\mathcal{C}$. Our goal is to determine the set of $\mathcal{C}$-composable rules and the operator of k-th composition on it.

A rule R *subsumes* a rule $\hat{R}$ ($R \leq \hat{R}$) if $\mathbb{I}_{\hat{R}} \subset \mathbb{I}_R$. Subsumption is Noetherian on the set of rules composable in a traditional Frege-Hilbert calculus (though not on the set of all possible rules). Thus, for each such rule there is a $\leq$-minimal rule subsuming it. An investigation of the set of $\leq$-minimal rules with the operation of composition followed by subsumption is left for future research.

2. Examples of Frege-Hilbert Calculi

The Propositional Calculus $\mathcal{C}_0$

$$A \to B \to A \tag{A1}$$

$$(A \to B \to C) \to (A \to B) \to A \to C \tag{A2}$$

$$(\neg A \to \neg B) \to B \to A \tag{A3}$$

$$\frac{A \quad A \to B}{B} \tag{R1}$$

The First Order Calculus $\mathcal{C}_1$

has the rules (A1), (A2), (A3), (R1), and the rules

$$\forall x F[x] \to F[t] \tag{A4}$$

$$\frac{A \to F[p]}{A \to \forall x F[x]} \qquad (p \notin A, F) \tag{R2}$$

In (A4), 't' denotes a ground term. In (R2), 'p' denotes a *parameter*, i.e., an auxiliary constant introduced by the calculus. The condition '$p \notin A, F$' means that the *eigenparameter* p of an instance of (R2) must not occur in A or in F.

The Propositional Calculus $\mathcal{C}_1'$

For a reason to become clear later, we use the symbol $\forall$ in this calculus to denote the unary propositional connective of identiy: $\forall\mathbf{true} = \mathbf{true}$ and $\forall\mathbf{false} = \mathbf{false}$. The calculus $\mathcal{C}_1'$ has the rules (A1), (A2), (A3), (R1), and the two rules

$$\forall F \to F \tag{A4$'$}$$

$$\frac{A \to F}{A \to \forall F} \tag{R2$'$}$$

3. Derivability in a Sound and Complete Frege-Hilbert Calculus

For $A \in \mathbb{S}$ let $\forall^{\mathrm{par}} A$ be the universal closure of the result of consistently and injectively replacing the parameters in A with variables not occuring in A (replacing each parameter p with $\iota(p)$ where ι is injective). For $S \subset \mathbb{S}$ let $\forall^{\mathrm{par}} S := \{\forall^{\mathrm{par}} A \mid A \in S\}$. For a calculus $\mathcal{C}$ let $S \vdash_{\mathcal{C}} B$ mean that B is in the closure of S under the rules of $\mathcal{C}$. Then all traditional sound and complete Frege-Hilbert calculi $\mathcal{C}$ such as $\mathcal{C}_0$, $\mathcal{C}_1$, and $\mathcal{C}_1'$ are strongly adequate: $S \vdash_{\mathcal{C}} B$ if and only if $\forall^{\mathrm{par}} S \models \forall^{\mathrm{par}} B$.

This strong soundness and completeness result gives us the set of all pairs $\langle S, B\rangle$ such that for some $A_1, \dots, A_n \in S$ the pair $\langle\langle A_1, \dots, A_n\rangle, B\rangle$ is an instance of a $\mathcal{C}$-composable rule. It is the set $\{\langle S, B\rangle \mid \forall^{\mathrm{par}} S \models \forall^{\mathrm{par}} B\}$. The result is, however, too coarse-grained to precisely determine the set of $\mathcal{C}$-composable rules.

4. A Variant of the Deduction Theorem

A parameter p is an *eigenparameter* of an instance I of a rule R if there is a constant a such that the result of replacing every occurrence of p in I with a is not an instance of R. The instance I is *compatible* with a formula D if the eigenparameters of I do not occur in D. For a rule R let $\vec{R}$ be the rule whose instances are the pairs $\langle\langle D \to A_1, \dots, D \to A_n\rangle, D \to B\rangle$ such that $\langle\langle A_1, \dots, A_n\rangle, B\rangle$ is compatible with D. A rule R is *good* for a calculus $\mathcal{C}$, or $\mathcal{C}$*-good*, if $\vec{R}$ is $\mathcal{C}$-composable. A calculus $\mathcal{C}$ is *good* if its rules are $\mathcal{C}$-good and invariant under permutations of parameters.

If (A1) and (A2) are $\mathcal{C}$-composable then (R1) and all eigenparameter-free axiom schemes of $\mathcal{C}$ are $\mathcal{C}$-good. The calculi $\mathcal{C}_0$, $\mathcal{C}_1$, and $\mathcal{C}_1'$ are good. Compositions of $\mathcal{C}$-good rules are $\mathcal{C}$-good. Thus rules composable in a good calculus $\mathcal{C}$ are $\mathcal{C}$-good.

Proposition 4.1. *Let $\mathcal{C}$ be a good calculus in which (A1), (A2), and (R1) are composable. Let $\phi\colon \{1, \dots, m\} \to \{0, \dots, n\}$. Let R be a $\mathcal{C}$-composable rule. Then there is an m-ary $\mathcal{C}$-composable rule $\hat{R}$ whose set of instances is the set of pairs $\langle\langle A_1, \dots, A_n\rangle, A_0 \to B\rangle$ such that $\langle\langle A_{\phi(1)}, \dots, A_{\phi(n)}\rangle, B\rangle$ is an instance of R which is compatible with A_0.*

5. Frege-Hilbert Calculi for Propositional Logic

A Frege-Hilbert calculus $\mathcal{C}$ for propositional logic uses propositional connectives as well as metavariables for formulas. For example, calculus $\mathcal{C}_0$ uses the connectives $\neg$ and $\to$ and the metavariables 'A', 'B', and 'C'. Let $\mathcal{T}$ be the free term algebra based on the propositional connectives of $\mathcal{C}$ as its function symbols and on some infinite superset of the set of metavariables of $\mathcal{C}$ as its variables. A *simple rule* is a rule of the form $\frac{\Phi_1 \dots \Phi_n}{\Psi}$ where $\Phi_1, \dots, \Phi_n, \Psi$ are terms of $\mathcal{T}$. The rules of traditional propositional Frege-Hilbert calculi are simple rules. Note that consistently and injectively replacing the metavariables in $\Phi_1, \dots, \Phi_n, \Psi$ with proposition symbols results in formulas $A_1, \dots, A_n, B$. We write $A_1, \dots, A_n \blacktriangleright B$ for $\frac{\Phi_1 \dots \Phi_n}{\Psi}$.

A composition of two simple rules can be computed using unification on the term algebra $\mathcal{T}$, much the same way as two first order Horn clauses are resolved using unification. It is again a simple rule. And a composition of two sound rules is again sound. Thus if $\mathcal{C}$ is a Frege-Hilbert calculus whose rules are sound and simple then every $\mathcal{C}$-composable rule is sound and simple. Unfortunately, it is not true in general that a sound simple rule is $\mathcal{C}$-composable, even if $\mathcal{C}$ is complete (as well as sound). A counterexample is the calculus obtained from $\mathcal{C}_0$ by replacing (R1) with $\frac{A \quad A \quad A \to B}{B}$. In it the rule (R1) is not composable.

Proposition 5.1. *Let $\mathcal{C}$ be a complete propositional Frege-Hilbert calculus such that (A1), (A2), and (R1) are $\mathcal{C}$-composable. Let $A_1, \dots, A_n \models B$. Then $A_1, \dots, A_n \blacktriangleright B$ is $\mathcal{C}$-composable.*

It is easy to show that $A_1, \dots, A_n \blacktriangleright B$ is subsumed by a $\mathcal{C}$-composable rule R. The harder part of the proof is to show that $A_1, \dots, A_n \blacktriangleright B$ subsumes R.

6. Frege-Hilbert Calculi for First Order Logic

Given a first order language $\mathcal{L}$, let $\mathcal{L}'$ be the propositional language whose proposition symbols are the predicate symbols of $\mathcal{L}$ and whose propositional connectives are the propositional connectives and quantifiers of $\mathcal{L}$. Quantifiers $\forall$ or $\exists$ of $\mathcal{L}$ denote the identity connective in $\mathcal{L}'$. For a first order formula F let F' be the formula of $\mathcal{L}'$ obtained from F by omitting arguments of predicate symbols and variables in quantifier prefixes. For example, if $F = \forall x(P(x) \to \neg\forall y Q(x, f(x, y)))$ then $F' = \forall(P \to \neg\forall Q)$. The operator $'$ is extended in the natural way to rules, to calculi (see the calculus $\mathcal{C}_1'$), and to derivations.

Proposition 6.1. *Let $\mathcal{C}$ be a first order Frege-Hilbert calculus. Then $'$ maps the set of derivations in $\mathcal{C}$ surjectively to the set of derivations in $\mathcal{C}'$. As a consequence, the set of instances of a composition of rules of $\mathcal{C}$ is nonempty if and only if the set of instances of the corresponding composition of rules of $\mathcal{C}'$ is nonempty.*

The significance of this proposition is that the propositional part of determining the composition of two rules and the part concerning the terms within the predicate symbols can be seperated from each other to a great extent. First the composition of the corresponding propositional rules of $\mathcal{C}'$ is computed which can be done by simple unification. If this succeeds then it is guaranteed that the composition of first order rules also represents a nonempty class of derivations. Then the terms and the constraints are computed which have to be imposed on rules in first order logic (see [3]). Constraints are eigenparameter conditions ($p \notin F$, $p \notin t$, $p \neq q$) or equations $F[s_1, \dots, s_m] = G[t_1, \dots, t_n]$ where $s_1, \dots, s_m, t_1, \dots, t_n$ are metavariables for ground terms or for parameters. For example, the axiom scheme $\forall x F[x] \to \neg\forall y \neg G[y] \quad (F[s] = G[t])$ is $\mathcal{C}_1$-composable.

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