Multirules in Classical First Order Predicate Logic

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In this paper we introduce the concept of multirules for a language $\mathcal{L}$ of classical first order predicate logic with function symbols and investigate the algebraic properties of the set of multirules. Let $\mathcal{S}$ be the set of sentences, i.e. closed formulas, of $\mathcal{L}$. In Frege-Hilbert calculi as well as in related calculi a proof of a sentence F of $\mathcal{S}$ may need to use formulas containing *eigen constants* or *parameters* which do not occur in F. For example, a formula $\forall x F[x]$ can be inferred from $F[p]$ only if the parameter p does not occur in $\forall x F[x]$. We assume that there is an infinite number of parameters and that they do not occur in $\mathcal{L}$. Let $\mathcal{L}_{\text{par}}$ be the language $\mathcal{L}$ augmented by the parameters as new constants, and let $\mathcal{S}_{\text{par}}$ be the set of sentences of $\mathcal{L}_{\text{par}}$. Then an n-ary rule of a Frege-Hilbert calculus defines a relation $R \subset \mathcal{S}_{\text{par}}^n \times \mathcal{S}_{\text{par}}$. For a sentence F of $\mathcal{S}_{\text{par}}$, let $\forall_{\text{par}} F$ denote the universal closure of the result of consistently replacing the parameters of F with pairwise distinct variables not occurring in F. For a finite or infinite sequence $\mathbf{F} = (F_1, F_2, \dots)$ of sentences of $\mathcal{S}_{\text{par}}$, let $\forall_{\text{par}} \mathbf{F}$ denote the set $\{\forall_{\text{par}} F_1, \forall_{\text{par}} F_2, \dots\}$. Then a rule defining a relation R is *valid* if for all $(\mathbf{F}, G) \in R$ it holds $\forall_{\text{par}} \mathbf{F} \models \forall_{\text{par}} G$. Rules of Frege-Hilbert calculi can be composed to form a new rule as has been shown in my earlier papers, and the composition of valid rules is again a valid rule.

Composition of rules is not a single binary operation on the set of rules. Rather, there are n ways to compose an m-ary rule with an n-ary rule. Each premise of the second rule can be chosen to unify with the conclusion of the first rule. Moreover, the composition of a rule with an axiom scheme (nullary rule) is undefined. For algebraic investigations it would be desirable to have a single operation which is defined for all rules. Therefore we introduce here the concept of a *multirule* which is similar to a tuple of rules, and we define the concept of a *product* of two multirules which has the desired properties. With the product operation, arbitrary compositions of rules and multirules can be constructed.

A *finite multirule* is a subset of $\mathcal{S}_{\text{par}}^m \times \mathcal{S}_{\text{par}}^n$ where m and n are nonnegative integers. A finite multirule represents an n-tuple $(R_1, \dots, R_n)$ of m-ary rules. If $F_1, \dots, F_m$ are the premises and each G_i $(i = 1, \dots, n)$ is the conclusion of an instance of the i-th rule R_i then the pair $((F_1, \dots, F_m), (G_1, \dots, G_n))$ is an element of the set R. However, composing two such multirules R_1 and R_2 with different values of arities m_1, n_1, m_2, n_2 would involve ugly case distinctions depending on the sign of $n_1 - m_2$ which make the investigation of algebraic properties complicated. Therefore we use a similar approach as the page description programming language Postscript does which operates on a pushdown stack. In our case this stack would be infinite. So we add an infinite number of premises and of conclusions to the multirule and require that the $(m+k)$-th premise is identical to the $(n+k)$-th conclusion of a rule instance for all $k \geq 1$.

Let $\mathbb{S}$ denote the set of infinite sequences $(F_1, F_2, \dots)$ of sentences of $\mathcal{S}_{\mathrm{par}}$. For $\mathbf{F} = (F_1, \dots, F_n) \in \mathcal{S}_{\mathrm{par}}^n$ and $\mathbf{H} = (H_1, H_2, \dots) \in \mathbb{S}$, let $\mathbf{FH}$ denote the sequence $(F_1, \dots, F_n, H_1, H_2, \dots)$ in $\mathbb{S}$. If R is a finite multirule then let $\bar{R}$ denote the set of pairs $(\mathbf{FH}, \mathbf{GH})$ such that $(\mathbf{F}, \mathbf{G}) \in R$ and $\mathbf{H} \in \mathbb{S}$. An *infinite multirule* is a set $\bar{R}$ such that R is a finite multirule. If R and S are two infinite multirules then the *product* RS of R and S is the set of pairs $(\mathbf{F}, \mathbf{H}) \in \mathbb{S} \times \mathbb{S}$ such that there is a $\mathbf{G} \in \mathbb{S}$ with $(\mathbf{F}, \mathbf{G}) \in R$ and $(\mathbf{G}, \mathbf{H}) \in S$. Let E be the identity function on $\mathbb{S}$. Then the set of infinite multirules together with the product operation is a monoid with neutral element E.

A finite or infinite multirule R is said to be *valid* if, for each pair $(\mathbf{F}, \mathbf{G}) \in R$, every model of $\forall_{\mathrm{par}}\mathbf{F}$ is also a model of $\forall_{\mathrm{par}}\mathbf{G}$. The product of two valid infinite multirules is again valid. So the set of valid infinite multirules is a submonoid of the monoid of all infinite multirules. The relation defined by a rule of a Frege-Hilbert calculus is just a finite multirule with exactly one conclusion, $R \subset \mathcal{S}_{\mathrm{par}}^n \times \mathcal{S}_{\mathrm{par}}^1$, and $\bar{R}$ is therefore an infinite multirule.

If $r_1, \dots, r_m, s_1, \dots, s_n \in \{1, \dots, k\}$ such that $\{s_1, \dots, s_n\} \subset \{r_1, \dots, r_m\}$ then the finite multirule $R = \{((F_{r_1}, \dots, F_{r_m}), (F_{s_1}, \dots, F_{s_n})) \mid F_1, \dots, F_k \in \mathcal{S}_{\mathrm{par}}\}$ is called a *finite multiprojection.* The corresponding infinite multirule $\bar{R}$ is called an *infinite multiprojection.* The set of infinite multiprojections is a submonoid of the monoid of valid infinite multirules. A *generalized composition* of two multirules R and S is a multirule $P_1 R P_2 S P_3$ where P_1, P_2 and P_3 are infinite multiprojections. The concept of generalized composition of multirules generalizes the concept of composition of rules as well as the concept of factorization of rules.

A finite or infinite multirule R is *bivalid* if, for all $(\mathbf{F}, \mathbf{G}) \in R$, every model of $\forall_{\mathrm{par}}\mathbf{F}$ is also a model of $\forall_{\mathrm{par}}\mathbf{G}$ and vice versa. The set of bivalid infinite multirules is a submonoid of the monoid of valid infinite multirules. A finite or infinite multirule R is *information preserving* if the following two propositions hold:

1. $(\mathbf{F}_1, \mathbf{G}) \in R$ and $(\mathbf{F}_2, \mathbf{G}) \in R$ implies $\mathbf{F}_1 = \mathbf{F}_2$.
2. $(\mathbf{F}, \mathbf{G}_1) \in R$ and $(\mathbf{F}, \mathbf{G}_2) \in R$ implies $\mathbf{G}_1 = \mathbf{G}_2$.

The set of information preserving infinite multirules is a submonoid of the set of all infinite multirules. The set of information preserving infinite multirules R with domain and range $\mathbb{S}$ is a submonoid thereof. Moreover, it is a group $\mathcal{G}$. The inverse of a multirule R of $\mathcal{G}$ is the multirule $R^{-1} = \{(\mathbf{G}, \mathbf{F}) \mid (\mathbf{F}, \mathbf{G}) \in R\}$. The set of bivalid multirules in $\mathcal{G}$ is a subgroup of the group $\mathcal{G}$.