Automated Deduction in Frege-Hilbert Calculi

Elmar Eder

University of Salzburg
Department of Computer Sciences

8th WSEAS International Conference on
Applied Computer and Applied Computational Science (ACACOS '09)
Hangzhou, May 20–22, 2009

Outline

Outline

Motivation

Main Objectives

- Automated deduction in classical first order logic
- Find short proofs
- Find them quickly
- Use a calculus which can express powerful proof principles

Frege-Hilbert Calculi

Gottlob Frege 1879: Begriffsschrift (concept language)

Axioms

$$A \rightarrow B \rightarrow A$$

$$(A \rightarrow B \rightarrow C) \rightarrow (A \rightarrow B) \rightarrow A \rightarrow C$$

$$(\neg A \rightarrow \neg B) \rightarrow B \rightarrow A$$

$$\forall x F \rightarrow F^x_t$$

Frege-Hilbert Calculi

Rules

$$\frac{A \qquad A \to B}{B} \qquad \text{modus ponens}$$

$$\frac{A \to F_p^x}{A \to \forall x F}$$

Constraint:
The parameter p must not occur in the conclusion $A \to \forall x F$.

Frege-Hilbert Calculi

A Proof Tree

$$\dfrac{P \rightarrow Q \rightarrow P \qquad \dfrac{P \rightarrow (Q \rightarrow P) \rightarrow P \qquad (P \rightarrow (Q \rightarrow P) \rightarrow P) \rightarrow (P \rightarrow Q \rightarrow P) \rightarrow P \rightarrow P}{(P \rightarrow Q \rightarrow P) \rightarrow P \rightarrow P}}{P \rightarrow P}$$

Frege-Hilbert Calculi

difficult to use for automated deduction

- Very inefficient to construct a proof by forward reasoning
- Better: backward reasoning
- But backward application of modus ponens not unique

$$\frac{A \qquad A \rightarrow B}{B}$$

- have to guess the cut formula A
- cut rule

Sequent Calculus

Gerhard Gentzen 1935

- Sequents of formulas instead of formulas
- Cut elimination theorem

Sequent calculus without cut rule allows

- analytic backward reasoning
- efficient proof search

Conventional Automated Deduction

Proof procedure

Construct the formulas of a proof one by one.

Calculi used

- J.A. Robinson 1965
  Resolution
- Calculi based on backward reasoning in the cut-free sequent calculus
  - Gerhard Gentzen 1935
    Sequent calculus without cut-rule
  - Wolfgang Bibel 1982
    Connection method
  - Evert W. Beth, Raymond M. Smullyan 1955–1971
    Tableau calculus

Cost of Cut Elimination

Theorem (R. Statman 1979, V.P. Orevkov 1982)

There is a sequence (F_n) *of formulas and a polynomial* p *such that each* F_n *has a proof of length* $\leq p(n)$ *in the full sequent calculus, but the shortest proof of* F_n *in the cut-free sequent calculus has length* $\geq \underbrace{2^{2^{\cdot^{\cdot^{\cdot^{2^2}}}}}}_{n \text{ times}}$.

Case Distinction

in a proof

- Case 1: Assume A.
  <Proof of B>
- Case 2: Assume $\neg A$.
  <Proof of B>

is a cut rule and equivalent to the general cut rule.

Cut and case distinction

are essential parts of human reasoning.

Outline

1 Introduction

2 Proof by Composition of Rules

Our Proof Procedure

Calculus

with cut rule (Frege-Hilbert calculus)

Construction of the proof

- Construct formula schemes and compositions of rules
- Later instantiate them to obtain the final proof.

Inference rule

Composition of rules of the Frege-Hilbert calculus

Our Proof Procedure

Calculus

with cut rule (Frege-Hilbert calculus)

Construction of the proof

- Construct formula schemes and compositions of rules
- Later instantiate them to obtain the final proof.

Inference rule

Composition of rules of the Frege-Hilbert calculus

Our Proof Procedure

Calculus

with cut rule (Frege-Hilbert calculus)

Construction of the proof

- Construct formula schemes and compositions of rules
- Later instantiate them to obtain the final proof.

Inference rule

Composition of rules of the Frege-Hilbert calculus

Our Proof Procedure

$\boxed{P \rightarrow P}$

Our Proof Procedure

$$\frac{A \qquad A \rightarrow P \rightarrow P}{P \rightarrow P}$$

Our Proof Procedure

$$\dfrac{A \qquad \dfrac{B \qquad B \to A \to P \to P}{A \to P \to P}}{P \to P}$$

Our Proof Procedure

$$\dfrac{P \to D \qquad \dfrac{P \to D \to P \qquad (P \to D \to P) \to (P \to D) \to P \to P}{P \to (D) \to P \to P}}{P \to P}$$

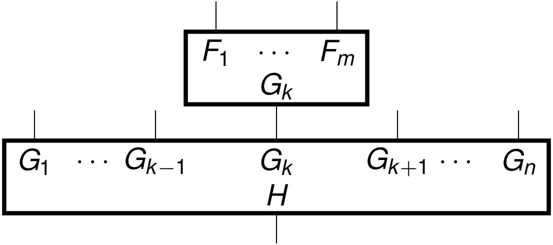
Our Proof Procedure

$$\dfrac{P \to C \to P \qquad \dfrac{P \to (C \to P) \to P \qquad (P \to (C \to P) \to P) \to (P \to C \to P) \to P \to P}{(P \to C \to P) \to P \to P}}{P \to P}$$

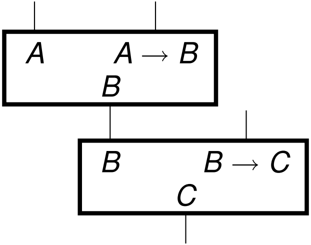
Our Proof Procedure

- A, B, C are meta-symbols standing for formulas
- formula schemes
- composition of rules
- reasoning in arbitrary directions

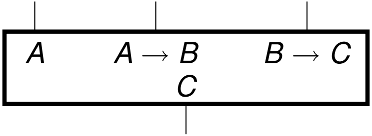
Composition of Rules



Composition of Rules



results in the rule



Composition of Rules

A proof using the rules

$$\forall x F \rightarrow F^x_t$$

and

$$\frac{A \rightarrow F^x_p}{A \rightarrow \forall x F}$$

$$\forall x P(x) \rightarrow P(p)$$
$$\forall x P(x) \rightarrow \forall y P(y).$$

Composition of Rules

$$\boxed{\forall x F \rightarrow F^x_t}$$

$$\boxed{\begin{array}{c} A \rightarrow F^x_p \\ A \rightarrow \forall x F \end{array}}$$

with the constraint $p \notin AF$

Composition of Rules

$$\boxed{\forall x F \rightarrow F^x_t}$$

$$\boxed{\begin{array}{c} A \rightarrow G^y_p \\ A \rightarrow \forall y G \end{array}}$$

with the constraint $p \notin AG$

Composition of Rules

$$\forall x F \rightarrow F^x_t$$

$$\forall x F \rightarrow G^y_p$$
$$\forall x F \rightarrow \forall y G$$

with the constraints $F^x_t = G^y_p$ and $p \notin FG$

Composition of Rules

$$\boxed{\forall x F \rightarrow \forall y G}$$

with the constraints $F_t^x = G_p^y$ and $p \notin FG$

Also yields a proof of

$$\forall x P(x) \rightarrow \forall y P(f(y))$$

The Most General Form of a Rule

$$\frac{\Phi_1 \quad \ldots \quad \Phi_n}{\Psi}$$

with a sequence $\mathcal{C}$ of constraints

Frege-Hilbert Calculus
Axioms

$$A \to B \to A$$

$$(A \to B \to C) \to (A \to B) \to A \to C$$

$$(\neg A \to \neg B) \to B \to A$$

$$\forall x F \to E \qquad \text{with}$$

constraint

$$E = F^x_t \, .$$

Frege-Hilbert Calculus
Rules

$$\frac{A \qquad A \rightarrow B}{B}$$

$$\frac{A \rightarrow E}{A \rightarrow \forall x F} \qquad \text{with}$$

constraints

$$E = F^x_p, \quad p \notin AF.$$

Rules

- Premises and conclusion are terms of a free term algebra.
- Composition by standardizing apart and unification
- Merge constraints

Conclusion

- Automated deduction feasible in calculi with cut
- Short proofs
- Expressiveness
- A student at our department is implementing a system.
- Future: sequent calculus