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Uniform Distribution with respect to Gaussian measure

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1 Introduction

Uniform distributed point sequences were extensively studied on the unit cube. In this note we want to generalize this concept to $\mathbb{R}^s$ with a weight function of Gaussian type. Also a Diaphony and a Weyl-type criterion is stated for such point sequences.

2 Uniform Distribution with respect to Gaussian measure

In the following we use the notation

$$\Phi\left(x\right):\mathbb{R}\rightarrow(0,1);\Phi(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x}e^{-\frac{t^2}{2}}dt=\frac{1}{2}\left(1+erf\left(\frac{x}{\sqrt{2}}\right)\right)$$

for the Gaussian distribution function with zero mean and variance equal to $\frac{1}{2}$ and with

$$erf\left(x\right):=\frac{2}{\sqrt{\pi}}\int_{0}^{x}e^{-t^2}dt.$$

The function

$$F\left(x\right):\mathbb{R}^s\rightarrow\left(0,1\right),F\left(x\right):=\prod_{j=1}^{s}\Phi\left(x^{(j)}\right)$$

is the distribution function of the s-dimensional standard normal distribution.

Definition 1. *We call a sequence $\{x_k\}_{k\geq 1}\in\mathbb{R}^s$ uniform distributed with respect to the measure $F\left(x\right)$ if*

$$\lim_{N\rightarrow\infty}\frac{1}{N}\sum_{k=1}^{N}\chi_I\left(x_k\right)=\frac{1}{\sqrt{2\pi}^s}\int_I e^{-\frac{|x|^2}{2}}dx \tag{1}$$

for all compact intervals $I\subset\mathbb{R}^s$.

Remark 1. In Definition 1 we have used only compact intervals I. It can be shown that if (1) is valid for all compact intervals then it is valid for all intervals J. Let J be an arbitrary interval. At first we consider a finite interval. Then we have

$$\left|\frac{1}{N}\sum_{k=1}^{N}\chi_J(x_k)-\frac{1}{\sqrt{2\pi}^s}\int_J e^{-\frac{|x|^2}{2}}dx\right|\leq\left|\frac{1}{N}\sum_{k=1}^{N}\chi_{\overline{J}}(x_k)-\frac{1}{\sqrt{2\pi}^s}\int_J e^{-\frac{|x|^2}{2}}dx\right|=$$

$$=\left|\frac{1}{N}\sum_{k=1}^{N}\chi_{\overline{J}}(x_k)-\frac{1}{\sqrt{2\pi}^s}\int_{\overline{J}} e^{-\frac{|x|^2}{2}}dx\right|$$

due to the continuity of the measure. Due to (1) and the compactness of $\overline{J}$ there is an $N_0(\varepsilon)$ with

$$\left|\frac{1}{N}\sum_{k=1}^{N}\chi_{\overline{J}}(x_k)-\frac{1}{\sqrt{2\pi}^s}\int_{\overline{J}} e^{-\frac{|x|^2}{2}}dx\right|<\varepsilon.$$

Now we consider the case of an infinite interval J. Let I be a compact interval with the property

$$\frac{1}{\sqrt{2\pi}^s}\int_{\mathbb{R}^s\setminus I} e^{-\frac{|x|^2}{2}}dx<\varepsilon. \tag{2}$$

Then we have

$$\left|\frac{1}{N}\sum_{k=1}^{N}\chi_J(x_k)-\frac{1}{\sqrt{2\pi}^s}\int_J e^{-\frac{|x|^2}{2}}dx\right|\leq\left|\frac{1}{N}\sum_{k=1}^{N}\chi_{I\cap J}(x_k)-\frac{1}{\sqrt{2\pi}^s}\int_{I\cap J} e^{-\frac{|x|^2}{2}}dx\right|+$$

$$+\frac{1}{N}\sum_{k=1}^{N}\chi_{J\setminus I}(x_k)+\frac{1}{\sqrt{2\pi}^s}\int_{J\setminus I} e^{-\frac{|x|^2}{2}}dx\leq$$

$$\leq\left|\frac{1}{N}\sum_{k=1}^{N}\chi_{\overline{I\cap J}}(x_k)-\frac{1}{\sqrt{2\pi}^s}\int_{\overline{I\cap J}} e^{-\frac{|x|^2}{2}}dx\right|+\frac{1}{N}\sum_{k=1}^{N}\chi_{\mathbb{R}^s\setminus I}(x_k)+\frac{1}{\sqrt{2\pi}^s}\int_{\mathbb{R}^s\setminus I} e^{-\frac{|x|^2}{2}}dx.$$

We have

$$\frac{1}{N}\sum_{k=1}^{N}\chi_{\mathbb{R}^s\setminus I}(x_k)=1-\frac{1}{N}\sum_{k=1}^{N}\chi_I(x_k)$$

Due to (1) there is an index $N_0(\varepsilon)$ with

$$\left|\frac{1}{N}\sum_{k=1}^{N}\chi_I(x_k)-\frac{1}{\sqrt{2\pi}^s}\int_I e^{-\frac{|x|^2}{2}}dx\right|<\varepsilon$$

for $N>N_0(\varepsilon)$ or equivalent

$$\frac{1}{\sqrt{2\pi}^s}\int_I e^{-\frac{|x|^2}{2}}dx-\varepsilon<\frac{1}{N}\sum_{k=1}^{N}\chi_I(x_k)<\frac{1}{\sqrt{2\pi}^s}\int_I e^{-\frac{|x|^2}{2}}dx+\varepsilon.$$

Therefor we get

$$1-\frac{1}{\sqrt{2\pi}^{s}}\int_{I}e^{-\frac{|x|^{2}}{2}}dx+\varepsilon>1-\frac{1}{N}\sum_{k=1}^{N}\chi_{I}\left(x_{k}\right)>1-\frac{1}{\sqrt{2\pi}^{s}}\int_{I}e^{-\frac{|x|^{2}}{2}}dx-\varepsilon.$$

So we get the estimation

$$\frac{1}{N}\sum_{k=1}^{N}\chi_{\mathbb{R}^{s}\setminus I}\left(x_{k}\right)<2\varepsilon.$$

for $N>N_{0}\left(\varepsilon\right)$. By (2) we have

$$\frac{1}{\sqrt{2\pi}^{s}}\int_{\mathbb{R}^{s}\setminus I}e^{-\frac{|x|^{2}}{2}}dx<\varepsilon$$

and due to (1) there is an index $N_{1}\left(\varepsilon\right)$ with

$$\left|\frac{1}{N}\sum_{k=1}^{N}\chi_{\overline{I\cap J}}\left(x_{k}\right)-\frac{1}{\sqrt{2\pi}^{s}}\int_{\overline{I\cap J}}e^{-\frac{|x|^{2}}{2}}dx\right|<\varepsilon$$

for $N>N_{1}\left(\varepsilon\right)$. So finally we have

$$\left|\frac{1}{N}\sum_{k=1}^{N}\chi_{J}\left(x_{k}\right)-\frac{1}{\sqrt{2\pi}^{s}}\int_{J}e^{-\frac{|x|^{2}}{2}}dx\right|<4\varepsilon$$

for $N>\max\left(N_{0}\left(\varepsilon\right),N_{1}\left(\varepsilon\right)\right)$.

Another criterion for a sequence being uniform distributed with respect to $F\left(x\right)$ is the following:

Theorem 1. *The point sequence $\left\{x_{k}\right\}_{k\geq1}\in\mathbb{R}^{s}$ is uniformly distributed with respect to $F\left(x\right)$ if and only if we have for all continuous functions $f\left(x\right)$ with compact support*

$$\lim_{N\to\infty}\frac{1}{N}\sum_{k=1}^{N}f\left(x_{k}\right)=\frac{1}{\sqrt{2\pi}^{s}}\int_{\mathbb{R}^{s}}f\left(x\right)e^{-\frac{|x|^{2}}{2}}dx. \tag{3}$$

Proof. Let $\left\{x_{k}\right\}_{k\geq1}\in\mathbb{R}^{s}$ be uniform distributed with respect to $F\left(x\right)$ and $f\left(x\right)$ be a continuous function with compact support. Then there are step functions g_{1},g_{2} with $g_{1}\leq f\leq g_{2}$ and

$$\int_{\mathbb{R}^{s}}\left(g_{2}-g_{1}\right)e^{-\frac{|x|^{2}}{2}}dx<\varepsilon.$$

Due to the uniform distribution we have (3) for step functions. Now we use the following (see [1])

Lemma 1. *Let* m_N $(N \in \mathbb{Z}, N \geq 0)$ *and* m *be positive functionals on some space* F *of real valued functions* $f : X \to \mathbb{R}$ $(X \neq \emptyset)$ *and let* $L \subseteq F$ *be the subspace of these functions* f *satisfying*

$$\lim_{N\to\infty} m_N(f) = m(f).$$

Suppose that $f \in F$ *has the property that for each* $\varepsilon > 0$ *there are functions* $g_1, g_2 \in L$ *with* $g_1 \leq f \leq g_2$ *and* $m(g_2) - m(g_1) < \varepsilon$. *Then we have* $f \in L$.

From this we have (3) for f. Now let (3) be valid for all continuous functions with compact support and let I be an arbitrary compact interval. Then there are continuous function g_1, g_2 with compact support with $g_1 \leq \chi_I \leq g_2$ and

$$\int_{\mathbb{R}^s} (g_2 - g_1)\, e^{-\frac{|x|^2}{2}} dx < \varepsilon.$$

Then by Lemma 1 we have (3) also for χ_I. □

It is also possible to use $f \in C_0^\infty$ in Theorem 1. So it can be reformulated in the following way:

Theorem 2. *The point sequence* $\{x_k\}_{k\geq 1} \in \mathbb{R}^s$ *is uniformly distributed with respect to* $F(x)$ *if and only if we have for all* $f(x) \in C_0^\infty$

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} f(x_k) = \frac{1}{\sqrt{2\pi}^s} \int_{\mathbb{R}^s} f(x)\, e^{-\frac{|x|^2}{2}} dx. \tag{4}$$

3 The reproducing kernel Hilbert space H_λ

Let $\lambda = (\lambda_1, \ldots, \lambda_s) \in \mathbb{R}^s$ with $|\lambda_i| < 1$. Let $H_n(x)$ be the n-th Hermite polynomial with the generating function

$$e^{2xt-t^2} = \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n$$

Let $\nu = (n_1, \ldots, n_s) \in \mathbb{N}_0^s$. It is well known that the system of functions

$$\left\{ \varphi_\nu\left(x^{(1)}, \ldots, x^{(s)}\right) := \prod_{j=1}^{s} e^{-\frac{\left(x^{(j)}\right)^2}{2}} \frac{H_{n_j}\left(x^{(j)}\right)}{\sqrt{2^{n_j} n_j! \sqrt{\pi}}} | \nu \in \mathbb{N}_0^s \right\}$$

forms a basis of the space $L^2(\mathbb{R}^s)$. For $\lambda \in \mathbb{R}^s$ and $\nu \in \mathbb{N}_0^s$ we use the abbreviation $\lambda^\nu := \lambda_1^{n_1} \ldots \lambda_s^{n_s}$. In the following we fix a λ and consider the space

$$H_\lambda := \left\{ f(x) : \mathbb{R}^s \to \mathbb{R} | f(x) := \sum_{\nu \in \mathbb{N}_0^s} a_\nu \lambda^\nu \varphi_\nu, \sum_{\nu\in\mathbb{N}_0^s} |a_\nu|^2 < \infty \right\}.$$

The following theorem can be shown:

Theorem 3. *Equipped with the scalar product defined by*

$$\langle \lambda^{\nu}\varphi_{\nu}, \lambda^{\mu}\varphi_{\mu}\rangle = \delta_{n_1 m_1}\dots\delta_{n_s m_s}$$

the space H_λ is a reproducing kernel Hilbert space with kernel

$$K : \mathbb{R}^s \times \mathbb{R}^s \to \mathbb{R}; K(x,y) =$$

$$= \frac{1}{\sqrt{\pi}^s}\prod_{j=1}^{s}\frac{1}{\sqrt{1-\lambda_j^4}}\exp\left(\frac{4x^{(j)}y^{(j)}\lambda_j^2 - \left(1+\lambda_j^4\right)\left(\left(x^{(j)}\right)^2 + \left(y^{(j)}\right)^2\right)}{2\left(1-\lambda_j^4\right)}\right).$$

Let $f(x) \in H_\lambda$. Then we have the following estimation for the integration error, which is analogous to the error estimation in [7]:

$$\left|\frac{1}{\sqrt{2\pi}^s}\int_{\mathbb{R}^s} f(x)\, e^{-\frac{|x|^2}{2}}dx - \frac{1}{N}\sum_{k=1}^{N} f(x_k)\right| \le$$

$$\le \|f\|_{H_\lambda}\cdot\left(\frac{1}{N^2}\sum_{k,l=1}^{N} K(x_k, x_l) - \frac{2}{N\sqrt{2\pi}^s}\sum_{k=1} e^{-\frac{|x_k|^2}{2}} + \frac{1}{\left(2\sqrt{\pi}\right)^s}\right)^{\frac{1}{2}}. \qquad (5)$$

Lemma 2. *Let $\{x_k\}_{k\ge 1} \in \mathbb{R}^s$ be a sequence uniform distributed with respect to the measure $F(x)$. Then we have*

$$\lim_{N\to\infty}\frac{1}{N}\sum_{k=1}^{N}\varphi_\nu(x_k) = \frac{1}{\sqrt{2\pi}^s}\int_{\mathbb{R}^s}\varphi_\nu(x)\, e^{-\frac{|x|^2}{2}}dx$$

for all $\nu \in \mathbb{N}_0^s$.

Proof. The functions $|\varphi_\nu(x)|$ are decreasing for $|x| \to \infty$. Now we consider a fix $\varphi_\nu(x)$. Let $\varepsilon > 0$. There is a compact interval $I(\nu,\varepsilon)$ with the property

$$|\varphi_\nu(x)| < \varepsilon$$

for $x \notin I(\nu,\varepsilon)$. There is a continuous function $f(x)$ with compact support $I(\nu,\varepsilon)$ with the property

$$\max_{x\in I(\nu,\varepsilon)} |f(x) - \varphi_\nu(x)| < \varepsilon.$$

Then we have

$$\left|\frac{1}{N}\sum_{k=1}^{N}\varphi_\nu(x_k) - \frac{1}{\sqrt{2\pi}^s}\int_{\mathbb{R}^s}\varphi_\nu(x)\, e^{-\frac{|x|^2}{2}}dx\right| \le$$

$$\le \left|\frac{1}{N}\sum_{k=1}^{N}\varphi_\nu(x_k)\,\chi_{\mathbb{R}^s\setminus I(\nu,\varepsilon)}(x_k)\right| + \left|\frac{1}{N}\sum_{k=1}^{N}\varphi_\nu(x_k)\,\chi_{I(\nu,\varepsilon)}(x_k) - \frac{1}{N}\sum_{k=1}^{N} f(x_k)\right| +$$

$$+\left|\frac{1}{N}\sum_{k=1}^{N} f\left(x_k\right)-\frac{1}{\sqrt{2\pi}^s}\int_{\mathbb{R}^s} f\left(x\right)e^{-\frac{|x|^2}{2}}dx\right|+$$

$$+\left|\frac{1}{\sqrt{2\pi}^s}\int_{\mathbb{R}^s} f\left(x\right)e^{-\frac{|x|^2}{2}}dx-\frac{1}{\sqrt{2\pi}^s}\int_{I(\nu,\varepsilon)}\varphi_\nu\left(x\right)e^{-\frac{|x|^2}{2}}dx\right|+$$

$$+\left|\frac{1}{\sqrt{2\pi}^s}\int_{\mathbb{R}^s\setminus I(\nu,\varepsilon)}\varphi_\nu\left(x\right)e^{-\frac{|x|^2}{2}}dx\right|\leq$$

$$\leq\varepsilon+\left|\frac{1}{N}\sum_{k=1}^{N}\varphi_\nu\left(x_k\right)\chi_{\mathbb{R}^s\setminus I}\left(x_k\right)\right|+\left|\frac{1}{N}\sum_{k=1}^{N} f\left(x_k\right)-\frac{1}{\sqrt{2\pi}^s}\int_{\mathbb{R}^s} f\left(x\right)e^{-\frac{|x|^2}{2}}dx\right|+$$
$$+2\varepsilon\leq$$

$$\leq 4\varepsilon+\left|\frac{1}{N}\sum_{k=1}^{N} f\left(x_k\right)-\frac{1}{\sqrt{2\pi}^s}\int_{\mathbb{R}^s} f\left(x\right)e^{-\frac{|x|^2}{2}}dx\right|.$$

Due to Theorem 1 there is an index $N\left(\varepsilon\right)$ with the property

$$\left|\frac{1}{N}\sum_{k=1}^{N} f\left(x_k\right)-\frac{1}{\sqrt{2\pi}^s}\int_{\mathbb{R}^s} f\left(x\right)e^{-\frac{|x|^2}{2}}dx\right|<\varepsilon$$

for $N>N\left(\varepsilon\right)$. So we get

$$\left|\frac{1}{N}\sum_{k=1}^{N}\varphi_\nu\left(x_k\right)-\frac{1}{\sqrt{2\pi}^s}\int_{\mathbb{R}^s}\varphi_\nu\left(x\right)e^{-\frac{|x|^2}{2}}dx\right|<5\varepsilon.$$

□

We denote the second factor in (5) by

$$F_N\left(x_n\right):=\left(\frac{1}{N^2}\sum_{k,l=1}^{N}K\left(x_k,x_l\right)-\frac{2}{N\sqrt{2\pi}^s}\sum_{k=1}^{N}e^{-\frac{|x_k|^2}{2}}+\frac{1}{\left(2\sqrt{\pi}\right)^s}\right)^{\frac{1}{2}}\qquad(6)$$

and can show the following

Theorem 4. *Let* $\{x_k\}_{k\geq 1}\in\mathbb{R}^s$ *be uniform distributed with respect to the measure* $F\left(x\right)$*. Then we have* $F_N\left(x_n\right)\to 0$ *for* $N\to\infty$*.*

Proof. The reproducing kernel has the series expansion

$$K\left(x,y\right)=\sum_{\nu\in\mathbb{N}_0^s}\varphi_\nu\left(x\right)\varphi_\nu\left(y\right)\lambda^{2\nu}.$$

With the property

$$\left|\varphi_\nu\left(x\right)\right|\leq 1$$

we have the uniform convergence of the series expansion:

$$\left|\sum_{\nu\in\mathbb{N}_0^s}\varphi_\nu(x)\,\varphi_\nu(y)\,\lambda^{2\nu}\right|\le\prod_{j=1}^s\left(\sum_{n=0}^\infty|\lambda_j|^{2n}\right)=\prod_{j=1}^s\frac{1}{1-\lambda_j^2}.$$

We have

$$\frac{1}{N^2}\sum_{k,l=1}^N K(x_k,x_l)=\frac{1}{N^2}\sum_{k,l=1}^N\sum_{\nu\in\mathbb{N}_0^s}\varphi_\nu(x_k)\,\varphi_\nu(x_l)\,\lambda^{2\nu}=$$

$$=\sum_{\nu\in\mathbb{N}_0^s}\lambda^{2\nu}\frac{1}{N^2}\sum_{k,l=1}^N\varphi_\nu(x_k)\,\varphi_\nu(x_l)=\sum_{\nu\in\mathbb{N}_0^s}\lambda^{2\nu}\left(\frac{1}{N}\sum_{k=1}^N\varphi_\nu(x_k)\right)^2.$$

By the uniform convergence of

$$\sum_{\nu\in\mathbb{N}_0^s}\lambda^{2\nu}\left(\frac{1}{N}\sum_{k=1}^N\varphi_\nu(x_k)\right)^2$$

we have $I\subset\mathbb{N}_0^s$ with

$$I:=\{(n_1,\dots,n_s)\,|0\le n_j\le M,j=0,\dots,s\}$$

with

$$\sum_{\nu\notin I}\lambda^{2\nu}\left(\frac{1}{N}\sum_{k=1}^N\varphi_\nu(x_k)\right)^2<\varepsilon.$$

So we have

$$F_N^2(x_n)=$$

$$=\sum_{\nu\in I}\lambda^{2\nu}\left(\frac{1}{N}\sum_{k=1}^N\varphi_\nu(x_k)\right)^2+\sum_{\nu\notin I}\lambda^{2\nu}\left(\frac{1}{N}\sum_{k=1}^N\varphi_\nu(x_k)\right)^2-$$

$$-\frac{2}{N\sqrt{2\pi}^s}\sum_{k=1}e^{-\frac{|x_k|^2}{2}}+\frac{1}{(2\sqrt{\pi})^s}.$$

By the fact that the point sequence is uniform distributed with respect to $F(x)$ we have an index $N_0(\varepsilon)$with

$$\left|\left(\frac{1}{N}\sum_{k=1}^N\frac{1}{\sqrt{\sqrt{\pi}}^s}\varphi_{(0,\dots,0)}(x_k)\right)^2-\frac{1}{(2\sqrt{\pi})^s}\right|<\varepsilon$$

for $N>N_0(\varepsilon)$ and we have an index $N_1(\varepsilon)$with

$$\left|\frac{1}{N}\sum_{k=1}^N\frac{e^{-\frac{|x_k|^2}{2}}}{\sqrt{2\pi}^s}-\frac{1}{(2\sqrt{\pi})^s}\right|<\varepsilon$$

and by Lemma 2 we have for all $\nu \in I \backslash (0, \ldots, 0)$ we have an index $N(\varepsilon, \nu)$ with

$$\left| \frac{1}{N} \sum_{k=1}^{N} \varphi_{\nu}(x_k) - \frac{1}{\sqrt{2\pi}^s} \int_{\mathbb{R}^s} \varphi_{\nu}(x) e^{-\frac{|x|^2}{2}} dx \right| < \varepsilon$$

for $N > N(\varepsilon, \nu)$. Then we have for $N > \max(\max_{\nu \in I} N(\varepsilon, \nu), N_0(\varepsilon), N_1(\varepsilon))$

$$F_N^2(x_n) \leq \left| \left(\frac{1}{N} \sum_{k=1}^{N} \frac{1}{\sqrt{\sqrt{\pi}}^s} \varphi_{(0,\ldots,0)}(x_k) \right)^2 - \frac{1}{(2\sqrt{\pi})^s} \right| +$$

$$+2 \left| \frac{1}{N} \sum_{k=1}^{N} \frac{e^{-\frac{|x_k|^2}{2}}}{\sqrt{2\pi}^s} - \frac{1}{(2\sqrt{\pi})^s} \right| + \sum_{\nu \in I \backslash (0,\ldots,0)} \lambda^{2\nu} \left(\frac{1}{N} \sum_{k=1}^{N} \varphi_{\nu}(x_k) \right)^2 +$$

$$+ \sum_{\nu \notin I} \lambda^{2\nu} \left(\frac{1}{N} \sum_{k=1}^{N} \varphi_{\nu}(x_k) \right)^2 <$$

$$< \varepsilon + 2\varepsilon + \sum_{\nu \in I \backslash (0,\ldots,0)} \lambda^{2\nu} \varepsilon^2 + \varepsilon \leq 4\varepsilon + \varepsilon^2 \prod_{j=1}^{s} \frac{1}{1 - \lambda_j^2}.$$

□

Remark 2. From the proof we see that it is possible to set $\lambda_j = \lambda_0$ with $|\lambda_0| < 1$ for $j = 1, \ldots, s$. If we have $F_N(x_n) \to 0$ for all $\lambda = (\lambda_0, \ldots, \lambda_0)$ then we have $F_N(x_n) \to 0$ for all $\lambda = (\lambda_1, \ldots, \lambda_s)$ with $|\lambda_j| < 1$: We set $\lambda_0 = \max(|\lambda_1|, \ldots, |\lambda_s|)$ and we get

$$\sum_{\nu \in I \backslash (0,\ldots,0)} \lambda^{2\nu} \left(\frac{1}{N} \sum_{k=1}^{N} \varphi_{\nu}(x_k) \right)^2 + \sum_{\nu \notin I} \lambda^{2\nu} \left(\frac{1}{N} \sum_{k=1}^{N} \varphi_{\nu}(x_k) \right)^2 \leq$$

$$\leq \sum_{\nu \in I \backslash (0,\ldots,0)} \lambda_0^{2\nu} \left(\frac{1}{N} \sum_{k=1}^{N} \varphi_{\nu}(x_k) \right)^2 + \sum_{\nu \notin I} \lambda_0^{2\nu} \left(\frac{1}{N} \sum_{k=1}^{N} \varphi_{\nu}(x_k) \right)^2 <$$

$$< \varepsilon^2 \sum_{\nu \in I \backslash (0,\ldots,0)} \lambda_0^{2\nu} + \varepsilon^2 \frac{1}{(1 - \lambda_0^2)^s}.$$

4 A Weyl criterion for a sequence uniform distributed with respect to $F(x)$

Let $f(x) \in C_0^{\infty}(\mathbb{R}^s)$. From the fact that the Riesz-means

$$(S_R^{\alpha} f)(x) = \sum_{n=0}^{\infty} \left(1 - \frac{2n+s}{R} \right)_+^{\alpha} \sum_{|\mu|=n} \varphi_{\mu}(x) \int_{\mathbb{R}^s} f(y) \varphi_{\mu}(y) dy$$

where

$$(x)_+ = \begin{cases} x & x > 0 \\ 0 & x \leq 0 \end{cases}$$

and $\alpha > \frac{s-1}{2}$ converge uniformly to $f(x) \in C_0^\infty$ for $R \to \infty$ (see [5]) we have for given $\varepsilon > 0$ an $R_0(\varepsilon) > 0$ with

$$\|(S_R^\alpha f) - f\|_\infty < \varepsilon$$

for $R > R_0(\varepsilon)$ on $\mathbb{R}^s$. Then we get

$$\left| \frac{1}{N} \sum_{k=1}^{N} f(x_k) - \frac{1}{\sqrt{2\pi}^s} \int_{\mathbb{R}^s} f(x) e^{-\frac{|x|^2}{2}} dx \right| \leq$$

$$\leq \left| \frac{1}{N} \sum_{k=1}^{N} f(x_k) - \frac{1}{N} \sum_{k=1}^{N} (S_R^\alpha f)(x_k) \right| +$$

$$+ \left| \frac{1}{N} \sum_{k=1}^{N} (S_R^\alpha f)(x_k) - \frac{1}{\sqrt{2\pi}^s} \int_{\mathbb{R}^s} (S_R^\alpha f)(x) e^{-\frac{|x|^2}{2}} dx \right| +$$

$$+ \left| \frac{1}{\sqrt{2\pi}^s} \int_{\mathbb{R}^s} (S_R^\alpha f)(x) e^{-\frac{|x|^2}{2}} dx - \frac{1}{\sqrt{2\pi}^s} \int_{\mathbb{R}^s} f(x) e^{-\frac{|x|^2}{2}} dx \right|.$$

Due to the uniform convergence we have

$$\left| \frac{1}{N} \sum_{k=1}^{N} f(x_k) - \frac{1}{N} \sum_{k=1}^{N} (S_R^\alpha f)(x_k) \right| < \varepsilon$$

for $R > R_0(\varepsilon)$. By Theorem 3.3.2 from [5] we have also convergence in L^2 so we get for the last term

$$\left| \frac{1}{\sqrt{2\pi}^s} \int_{\mathbb{R}^s} (S_R^\alpha f)(x) e^{-\frac{|x|^2}{2}} dx - \frac{1}{\sqrt{2\pi}^s} \int_{\mathbb{R}^s} f(x) e^{-\frac{|x|^2}{2}} dx \right| \leq$$

$$\leq \frac{1}{\sqrt{2\pi}^s} \left(\int_{\mathbb{R}^s} (S_R^\alpha f - f)^2 dx \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^s} e^{-|x|^2} dx \right)^{\frac{1}{2}} < \frac{\varepsilon}{\sqrt{2}^s}$$

for $R > R_1(\varepsilon)$. In the second term we consider the case $R > \max(R_0, R_1)$. The Riesz-means are finite linear combinations of the Hermite functions:

$$(S_R^\alpha f)(x) = \sum_{\mu \in I \subset \mathbb{N}_0^s} c_\mu \varphi_\mu(x).$$

Now suppose that we have for all Hermite functions the following:

$$\lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} \varphi_\nu(x_k) = \frac{1}{\sqrt{2\pi}^s} \int_{\mathbb{R}^s} \varphi_\nu(x) e^{-\frac{|x|^2}{2}} dx \qquad (7)$$

for all $\nu \in \mathbb{N}_0^s$. Then we have for all $\mu \in I$ an index $N_0(\varepsilon, \mu)$ with the property

$$\left| \frac{1}{N} \sum_{k=1}^{N} \varphi_\nu(x_k) - \frac{1}{\sqrt{2\pi}^s} \int_{\mathbb{R}^s} \varphi_\nu(x) e^{-|x|^2} dx \right| < \varepsilon$$

for $N > N_0(\varepsilon, \mu)$ and for $N > \max_{\mu \in I} N_0(\varepsilon, \mu)$ we have

$$\left| \frac{1}{N} \sum_{k=1}^{N} (S_R^\alpha f)(x_k) - \frac{1}{\sqrt{2\pi}^s} \int_{\mathbb{R}^s} (S_R^\alpha f)(x) e^{-\frac{|x|^2}{2}} dx \right| \leq$$

$$\leq \sum_{m=0}^{\infty} \left(1 - \frac{2m+s}{R}\right)_+^\alpha \left[\sum_{|\mu|=m} |c_\mu| \times \right.$$

$$\left. \times \left| \frac{1}{N} \sum_{k=1} \varphi_\mu(x_k) - \frac{1}{\sqrt{2\pi}^s} \int_{\mathbb{R}^s} \varphi_\mu(x) e^{-\frac{|x|^2}{2}} dx \right| \right] <$$

$$< \varepsilon \sum_{m=0}^{\infty} \left(1 - \frac{2m+s}{R}\right)_+^\alpha \sum_{|\mu|=m} |c_\mu|$$

with the coefficients

$$c_\mu = \int_{\mathbb{R}^s} f(x) \varphi_\mu(x) dx.$$

Therefor we get the estimation

$$\varepsilon \sum_{m=0}^{\infty} \left(1 - \frac{2m+s}{R}\right)_+^\alpha \sum_{|\mu|=m} |c_\mu| \leq \varepsilon \left(\sum_{\mu \in \mathbb{N}_0^s} |c_\mu|^2 \right)^{\frac{1}{2}} = \varepsilon \|f\|_2.$$

Summing up we get

$$\left| \frac{1}{N} \sum_{k=1}^{N} f(x_k) - \frac{1}{\sqrt{2\pi}^s} \int_{\mathbb{R}^s} f(x) e^{-\frac{|x|^2}{2}} dx \right| < \varepsilon + \varepsilon \|f\|_2 + \frac{\varepsilon}{\sqrt{2\sqrt{\pi}}^s}$$

for $R > \max(R_0(\varepsilon), R_1(\varepsilon))$and $N > \max_{\mu \in I} N_0(\varepsilon, \mu)$. So assumption (7) is sufficient to have

$$\lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} f(x_k) = \frac{1}{\sqrt{2\pi}^s} \int_{\mathbb{R}^s} f(x) e^{-\frac{|x|^2}{2}} dx$$

for $f \in C_0^\infty(\mathbb{R}^s)$. So we have the following

Theorem 5. *The sequence $\{x_k\}_{k \geq 1} \in \mathbb{R}^s$ is uniform distributed with respect to $F(x)$ if and only if we have*

$$\lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} \varphi_\nu(x_k) = \frac{1}{\sqrt{2\pi}^s} \int_{\mathbb{R}^s} \varphi_\nu(x) e^{-\frac{|x|^2}{2}} dx \tag{8}$$

for all $\nu \in \mathbb{N}_0^s$.

Proof. From the computations above the criterion (8) is sufficient to have

$$\lim_{N\to\infty}\frac{1}{N}\sum_{k=1}^{N} f\left(x_k\right)=\frac{1}{\sqrt{2\pi}^s}\int_{\mathbb{R}^s} f\left(x\right)e^{-\frac{|x|^2}{2}}dx$$

for all $f\in C_0^\infty\left(\mathbb{R}^s\right)$ which is equivalent to the uniform distribution of the sequence by Theorem 2. If the sequence is uniform distributed then we have the property (8) by Lemma 2. □

Remark 3. From Theorem 4 we can deduce another criterion for uniform distribution with respect to $F\left(x\right)$:

Theorem 6. *The sequence* $\left\{x_k\right\}_{k\geq 1}\in\mathbb{R}^s$ *is uniform distributed with respect to* $F\left(x\right)$ *if and only if we have have* $F_N\left(x_n\right)\to 0$ *for* $N\to\infty$.

Proof. From Theorem 4 we have $F_N\left(x_n\right)\to 0$ for $N\to\infty$ for a point sequence $\left\{x_k\right\}_{k\geq 1}\in\mathbb{R}^s$ uniform distributed with respect to $F\left(x\right)$. Now let $F_N\left(x_n\right)\to 0$ for $N\to\infty$. Due to the fact that $\varphi_\nu\in H_\lambda$ for all $\nu\in\mathbb{N}_0^s$ the error estimation (5) is valid. With

$$\left\|\varphi_\nu\right\|_{H_\lambda}=\frac{1}{\lambda^\nu}$$

we have

$$\left|\frac{1}{N}\sum_{k=1}^{N}\varphi_\nu\left(x_k\right)-\frac{1}{\sqrt{2\pi}^s}\int_{\mathbb{R}^s}\varphi_\nu\left(x\right)e^{-|x|^2}dx\right|\leq\frac{1}{\lambda^\nu}F_N\left(x_n\right)\to 0$$

for all $\nu\in\mathbb{N}_0^s$ and therefor by Theorem 5 we have uniform distribution of the point sequence. □

5 Numerical results

In the following we present some numerical results using the built-in random number generator of R [4] and the Zinterhof sequence [6]. We compute (6) and the integration error of a test function $f(x_1, \ldots, x_s)$. Due to the scaling of (6) we compute $\pi^{\frac{s}{4}} F_N(x_n)$. As parameter λ we choose $\lambda_j = 0.5$ for $j = 1, \ldots, s$. In the following we denote the number of points by N and the dimension by s.

1. The R-built in random generator: We use the point sequence $x_n = (x_{n,1}, \ldots, x_{n,s})$ where each vector is generated by the built-in random generator with normal distribution $N(0, I)$.

s	N	$\pi^{\frac{s}{4}} F_N(x_n)$
1	8192	0.005748381
1	65536	0.001103242
1	106496	0.002917923
5	8192	0.004736215
5	65536	0.001080732
5	106496	0.001352613
10	8192	0.00188253
10	65536	0.0005729474
10	106496	0.0005028543
70	8192	4.513478e-08
70	65536	1.083406e-08

2. Zinterhof sequence: Set $\theta = \left(\exp(1), \exp\left(\frac{1}{2}\right), \ldots, \exp\left(\frac{1}{s}\right)\right)$. Then we use the sequence $x_n = \left(\Phi^{-1}(\{n\theta_1\}), \ldots, \Phi^{-1}(\{n\theta_s\})\right)$ with $\{x\} = x \bmod 1$.

s	N	$\pi^{\frac{s}{4}} F_N(x_n)$
1	8192	8.214825e-05
1	65536	9.455125e-05
1	106496	9.058794e-09
5	8192	0.0004311535
5	65536	0.0001032745
5	106496	8.713051e-05
10	8192	0.0008466832
10	65536	0.0003279167
10	106496	0.0001804514
70	8192	5.463154e-08
70	65536	1.357898e-08

Conclusion: The built-in random generator and the Zinterhof sequence show comparable results. For small s the Zinterhof sequence is much better than the random vector sequence. For higher dimensionality this advantage seems to decrease which is also founded by the used inversion method. In case of the Zinterhof sequence a decrease of $\pi^{\frac{s}{4}} F_N(x_n)$ by increasing N is visible which is not the case for the random vector sequence (especially for small dimensions).

Now we have a look to the behavior of $\pi^{\frac{s}{4}} F_N(x_n)$ under increase of the dimension. In the table below we have values for $\pi^{\frac{s}{4}} F_N(x_n)$, $N = 16384$ and various dimensions. The point sequence is generated by the R built-in random generator.

s	$\pi^{\frac{s}{4}} F_N(x_n)$
1	0.001756766
3	0.00204581
7	0.002421785
12	0.0009855352
17	0.0003569184
22	0.0001435636
35	1.555285e-05
60	1.256569e-07
90	2.661663e-10

This shows that $\pi^{\frac{s}{4}} F_N(x_n)$ is still not properly normalized. This problem is also present in the ordinary L^2-discrepancy. For the L^2-discrepancy it is discussed in [3], for reproducing kernel Hilbert spaces see [2].

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