Algebraic Properties of Rules of Frege-Hilbert Calculi

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Elmar Eder*

Abstract. *Whereas resolution and calculi based on backward cut-free sequent calculi are most widely used for automated deduction in classical first order logic, the use of Frege-Hilbert calculi greatly reduces the size of proofs of some formulas. However, the search for a proof in a Frege-Hilbert calculus cannot be done efficiently in the usual step-by-step manner. Rather, compositions and factors of rules must be constructed. In this paper, some of the problems and results of investigating the set of rules and its algebraic properties, are discussed.*

1. Introduction

The first formalization in the history of predicate logic was Gottlob Frege's Begriffsschrift [7] published in 1879. In his paper Frege also gives a proof calculus for the Begriffsschrift. It turned out that Frege's calculus was too strong. In fact, it was contradictory as Bertrand Russel pointed out in a letter to Frege. Whitehead and Russel showed in their logical formalism of Principia Mathematica [16] how to avoid the contradictions in the calculus by restricting the language of logic to typed logic. Frege's calculus has some features like definitions and the substitution rule which are convenient for the formulation and proof of propositions and also can make formulas and proofs considerably shorter. But these features can be omitted from the calculus without weakening the expressiveness of the language or of the proof calculus. David Hilbert and many other logicians set up and studied such calculi which are similar to Frege's calculus, but are reduced to the bare minimum. Such calculi are today called *Frege-Hilbert calculi*. Whereas calculi set up by Hilbert and others for classical first order predicate logic are complete (Kurt Gödel's completeness theorem of 1930), Kurt Gödel showed in 1931 [9] that there is no complete calculus for a logic of order higher than one. In the present paper, we consider only Frege-Hilbert calculi for classical propositional and first order predicate logic.

A Frege-Hilbert calculus consists of a set of *rules* of logic reasoning. Each rule has the form $\frac{\Phi_1 \dots \Phi_n}{\Psi}$, possibly with a condition restricting its applicability. Each of the *premises* $\Phi_1, \dots, \Phi_n$ of the rule and the *conclusion* Ψ of the rule are *formula schemes*. A rule $\frac{}{\Psi}$ with no premises is called an *axiom scheme* and written Ψ for short. A formula scheme is a string of symbols which is used to denote a formula. It may contain meta-symbols denoting formulas, variables, terms, and parameters. Here is an example of a Frege-Hilbert calculus where "A", "B", "C", and "F" are meta-symbols denoting formulas, "x" is a meta-symbol denoting a variable, "t" is a meta-symbol denoting a term, and "p" is a meta-symbol denoting a parameter. "$F[x\backslash t]$" denotes the result of replacing every free occurrence of the variable x in the formula F with the term t.

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A Frege-Hilbert calculus $\mathcal{C}$

$$A \to B \to A \tag{A1}$$

$$(A \to B \to C) \to (A \to B) \to A \to C \tag{A2}$$

$$(\neg A \to \neg B) \to B \to A \tag{A3}$$

$$\forall x F \to F[x \backslash t] \tag{A4}$$

$$\frac{A \qquad A \to B}{B} \tag{R1}$$

$$\frac{A \to F[x \backslash p]}{A \to \forall x F} \quad \text{with parameter condition} \tag{R2}$$

Rule (R2) may only be applied if the *eigenparameter* p does not occur in the conclusion $A \to \forall x F$. Also, in a derivation of a formula from a set S of formulas, p must not occur in any formula of S.

Despite the restrictions and simplifications of present-day Frege-Hilbert calculi as compared to Frege's original calculus, Frege-Hilbert calculi are still very powerful in expressing proof concepts. In fact, the great amount of freedom they allow in constructing a proof makes it very difficult to use them for automated deduction, since every node of the search tree has an infinite branching factor. Doing forward reasoning, i.e., starting from the axioms and applying the rules until the assertion has been proved, is inefficient because there are infinitely many axioms. Backward reasoning, i.e., starting from the assertion and applying the rules in a backward direction to arrive at axioms, seems to be a better approach. But even this is inefficient. The reason is that a Frege-Hilbert calculus has a *cut rule* such as the Modus Ponens (R1) of calculus $\mathcal{C}$. A cut rule is a rule which has a meta-symbol for a formula occurring in one or more of its premises but not in its conclusion. In (R1) it is the symbol "A". If we do backward reasoning in $\mathcal{C}$ and we want to prove a formula B by applying (R1) in backward direction then we have to find a formula A such that A and $A \to B$ are provable. But there is an infinite number of formulas we can choose from for A. There is no way to find the right one at this stage of the process of construction of a proof.

Therefore most automatic theorem provers use much more restricted calculi such as Robinson's resolution calculus [12] or calculi based on backward reasoning in cut-free versions of Gentzen's sequent calculus [8]. The latter include, for example, Bibel's connection method (see [3]) and various versions of Beth's and Smullyan's tableau calculus [1, 2, 13]. The full sequent calculus has a cut rule which makes automatic reasoning in it inefficient. But Gentzen has proved in [8] that applications of the cut rule can be eliminated from any proof in his calculus. Therefore, the sequent calculus without the cut rule is still complete. Backward reasoning in it can be used and has been used as a basis for calculi for automated reasoning.

However, Statman [14] and Orevkov [11] have proved that cut elimination necessarily extremely blows up the size of shortest proofs of some formulas. These formulas are not just academic examples. Rather, they occur naturally as theorems in established disciplines of mathematics, e.g. in the theory of combinatorial logic. So it seems that, even for everyday automated deduction, we must find a way to use calculi with a cut rule.

I have shown in [5] that, by a sort of unification on the level of formulas rather than terms, the infinite branching of the search tree can be avoided in backward reasoning in calculi with cut. One way to learn more about backward reasoning, forward reasoning, and mixed forward and backward reasoning in such calculi, is to study compositions of rules. The simplest calculi with cut are Frege-Hilbert calculi. Once composition of rules is understood for Frege-Hilbert calculi, it should not be too difficult to understand composition of rules in other calculi such as sequent calculi with cut. This paper is meant to show some of the problems and results in trying to understand the composition of rules in Frege-Hilbert calculi.

2. The formula concept

In the introduction, the (axiom schemes and) rules of a Frege-Hilbert calculus $\mathcal{C}$ have been given using meta-symbols for formulas, for variables, for terms, and for parameters. In order for such rules to make sense, it is necessary to say what we mean by a formula, variable, term, or parameter. In the literature of classical first order predicate logic, there is no general agreement about the concept of a formula. One aspect upon which there is disagreement in literature is whether a formula F is to be regarded as identical to a formula obtained from F by renaming free occurences of variables. For example, should the formula $\forall x P(x)$ and the formula $\forall y P(y)$ be identified with each other?

Most authors do not identify these two formulas. The most frequently used formula concept considers a formula as a string of symbols. Concepts using other data structures, e.g. trees, are isomorphic to it. So, we shall here only consider strings of symbols. Strings of symbols which are formulas in this sense, will be called *formula strings*. As an example, here is an inductive definition of the concept of formula strings based on the set of logical symbols $\{\neg, \rightarrow, \forall\}$ used for the calculus $\mathcal{C}$.

1. If P is an n-ary predicate symbol and $t_1, \dots, t_n$ are terms then the string $P(t_1, \dots, t_n)$ is a formula string.
2. If A is a formula string then so is the string $\neg A$.
3. If A and B are formula strings then so is the string $(A \rightarrow B)$.
4. If x is a variable and F is a formula string then the string $\forall x F$ is a formula string.

The definition is similar for sets of logical symbols other than $\{\neg, \rightarrow, \forall\}$.

Note that, for the set of logical symbols $\{\neg, \wedge, \vee, \forall, \exists\}$, the string $\forall x \exists x P(x)$ is a formula string. In it, the occurrence of the variable x in $P(x)$ is in the scope of two quantifier prefixes $\forall x$ and $\exists x$. The innermost (i.e., rightmost) of these two quantifier prefixes, namely $\exists x$, is considered to bind the occurrence of x in $P(x)$. A formula string like this is rather confusing to read. However, you can rename its bound variables so that no variable is quantified twice. Just rename any occurrence of a bound variable according to the quantifier occurrence binding it. In the above example, rename variables bound by $\forall$ by u and variables bound by $\exists$ by v, and you get a formula string $\forall u \exists v P(v)$ which is semantically equivalent to the original formula string.

Therefore, there are different concepts of formulas according to whether a variable is allowed to be in the scope of more than one quantifier occurrence for that variable.

1. According to the most liberal concept of formulas, any formula string is formula.

2. A more restricted concept requires that an occurrence of a variable in a formula must not be in the scope of two quantifier prefixes for that same variable.

3. The most restrictive policy in this respect would not allow two distinct quantifier occurrences quantifying the same variable within a formula.

All of these formula concepts are equivalent to each other with respect to their expressive power. By renaming of bound variables, it can always be guaranteed that no variable is quantified twice. However, 3. is not suitable for Frege-Hilbert calculi since, e.g., $\forall x P(x) \to U \to \forall x P(x)$ would not be a formula. For the calculus $\mathcal{C}$ this would mean that the formula A in (A1) has to be quantifier-free. Otherwise, $A \to B \to A$ would not be a formula according to 3. Likewise, F in (A4) would have to be quantifier-free. This would imply that the formula $\forall x \forall y P(x, y) \to P(a, b)$ would not be provable in $\mathcal{C}$, and thus $\mathcal{C}$ would not be complete. Likewise other Frege-Hilbert calculi would be incomplete if a formulation of the concept of formulas according to 3. was used. Note, however, that sequent calculi and tableau calculi can cope with such a notion of formula.

For formula concepts according to 1. or 2., the corresponding Frege-Hilbert calculi are equivalent to each other in the following sense. If F is a formula according to 2. then it is also a formula according to 1. Then, any proof of F in a Frege-Hilbert calculus according to 2. is also a proof in the corresponding Frege-Hilbert calculus according to 1. And any proof of F in a Frege-Hilbert calculus according to 1. can be transformed to a proof of F in the corresponding Frege-Hilbert calculus according to 2. The transformation is stepwise and does not increase the number of steps or the complexity of the proof. Moreover, if F is a formula according to 1. then F can be transformed to a semantically equivalent formula F' according to 2. just by renaming of bound variables. Moreover, every proof of F in a Frege-Hilbert calculus according to 1. can be stepwise transformed to a proof of F' in the corresponding Frege-Hilbert calculus according to 2., and vice versa.

So it seems that, if formulas are defined to be formula strings subject to some restriction, all restrictions to formula strings which are widely used in literature, either are not suitable for Frege-Hilbert calculi, or Frege-Hilbert calculi using them behave the same as Frege-Hilbert calculi using the concept of (unrestricted) formula strings. There seems to be no point in restricting the concept of formula strings in the study of Frege-Hilbert calculi.

Some authors do, however, identify two formula strings which are obtained from each other by renaming of variables. Again, there are several ways to do this. One way is to define an equivalence relation on the set of formula strings. Two formula strings F and G are equivalent to each other if and only if G is obtained from F by replacing bound occurrences of variables in F by variables in such a way that if the variable at a place κ in F is bound by the quantifier at a place λ in F then the variable at place κ in G is bound by the quantifier at place λ in G. We speak of "equivalence modulo renaming of bound variables". Then, a formula is defined to be an equivalence class of formula strings. Another way is to choose a representative of each equivalence class, a kind of normal form for all formula strings of the equivalence class. For example, let the variables of the language of logic be brought into an order $x_1, x_2, x_3, \ldots$. Then a formula string with n quantifier occurrences may be considered to be in normal form if its quantifier occurrences quantify the variables $x_1, \ldots, x_n$ in this order. Then a formula would be defined as a formula string in normal form. Of course, a transformation to normal form would have to be performed after every construction of a formula string from sub-formula strings using propositional connectives or quantifiers. Likewise, the definition of $F[x\backslash t]$ has to involve transformation to normal form. Usually authors just say they identify formulas which are obtained from each other by renaming of bound variables. They do not state which way they choose to do this. The reason for this is that all ways to achieve this lead to isomorphic concepts of formulas and are therefore equivalent to each other.

As has been shown in [6], a formula modulo renaming of variables can be transformed to a formula string such that a Frege-Hilbert proof of a formula modulo renaming of variables is transformed to a Frege-Hilbert proof of the corresponding formula string. And, of course, identification modulo renaming of bound variables maps Frege-Hilbert proofs to Frege-Hilbert proofs. However, since these transformations are not inverses of each other, not all formulas which have short proofs with identification modulo renaming of bound variables, also have short proofs without this identification. A counterexample is the formula $\forall x P(x) \to U \to \forall y P(y)$ given in [6]. Thus, Frege-Hilbert calculi with and without identification modulo renaming of bound variables are equivalent with respect to their expressive power, but still they seem not to be isomorphic to each other in any reasonable sense of the word. So it should be interesting to investigate both kinds of calculi.

Logicians often consider classical first order predicate logic without function symbols, which is as expressive as classical first oder logic with function symbols. For automated deduction it is useful to have function symbols since they often allow to express propositions more concisely. Moreover, Skolemization using function symbols together with unification is a powerful tool for automatic deduction.

Another point of disagreement among authors is about variables in formulas. Some authors distinguish between free and bound variables. If we allow function symbols, we do not need this distinction, since constants (nullary function symbols) can play the role of free variables. Some authors allow variables to occur free in premises and conclusions of rule instances. They have to add an extra condition on rules such as (A4) stating that the varialbe x must not occur free in the formula F within the scope of a quantifier binding a variable which occurs in the term t. If we require that all premises and conclusions of rule instances are closed formulas then this condition is automatically fulfilled.

Parameters play the role of auxiliary constants introduced by rules of the calculus such as (R2). In its premise occurs a parameter p. For the purpose of proving a formula, it suffices to consider parameters as constants of the given language. This does not work, however, if the calculus is used to derive a formula from an infinite set S of formulas. The reason is that the parameter condition requires that the parameter p must not occur in S. Now, if every constant of the logical language occurrs in some formula of S then there is no constant left which obeys the parameter condition. Thus, the calculus becomes incomplete for deriving a formula from an infinite set of formulas. Therefore, it is better to enlarge the set of constants by a countable infinite number of auxiliary constants, called *parameters*. Now, if a formula F is to be derived from a set S of formulas, then F and the elements of S must be closed formulas of the original logical language. However, the premises and conclusions of rule instances are closed formulas of the enlarged logical language. Some authors use variables rather than auxiliary constants as parameters.

In this paper, we mean by a *formula* a formula string with or without parameters. A *sentence* is a closed formula. The set of sentences is denoted $\mathfrak{S}^{\mathrm{par}}$ and the set of parameter-free sentences is denoted $\mathfrak{S}$. Thus, the calculus is used only to prove a sentence of $\mathfrak{S}$ or to derive it from a set of sentences of $\mathfrak{S}$. But the derivations use sentences of $\mathfrak{S}^{\mathrm{par}}$. In fact, the set of formulas which underlies the calculus is the set $\mathfrak{S}^{\mathrm{par}}$. So, an investigation of algebraic properties of a Frege-Hilbert calculus will deal with the set $\mathfrak{S}^{\mathrm{par}}$ rather than with the set $\mathfrak{S}$.

3. Rules and deduction relations

For the following concepts we need a few notations. The set $\{0, 1, 2, \dots\}$ of natural numbers is denoted $\mathbb{N}$. A tuple of objects $z_1, \dots, z_n$ is denoted $\langle z_1, \dots, z_n \rangle$. For a set Z, the set $\bigcup_{n \in \mathbb{N}} Z^n$ of tuples of elements of Z is denoted Z^*.

An *instance* of a rule $\frac{\Phi_1 \dots \Phi_n}{\Psi}$ is a pair $\langle \langle F_1, \dots, F_n \rangle, G \rangle$ where $F_1, \dots, F_n, G$ are sentences obtained from $\Phi_1, \dots, \Phi_n, \Psi$, respectively, by consistently replacing meta-symbols for formulas, variables, terms, and parameters, with actual formulas, variables, terms, and parameters, respectively. If there is a condition attached to a rule, the condition is required to hold for the rule instance. The formulas $F_1, \dots, F_n$ are called the *premises* of the rule instance, and G is called the *conclusion* of the rule instance. For example, if F and G are sentences then $\langle \langle F, F \to G \rangle, G \rangle$ is an instance of the modus ponens rule (R1) with premises F and $F \to G$ and conclusion G.

A *deduction relation* (on $\mathfrak{S}^{\mathrm{par}}$) is a subset of $\mathfrak{S}^{\mathrm{par}*} \times \mathfrak{S}^{\mathrm{par}}$. If $\rhd$ is a deduction relation then an *instance* of $\rhd$, for short $\rhd$*-instance*, is an element of $\rhd$. If $\langle \langle F_1, \dots, F_n \rangle, G \rangle$ is a $\rhd$-instance then we write $\langle F_1, \dots, F_n \rangle \rhd G$. The sentences $F_1, \dots, F_n$ are called the *premises* of the $\rhd$-instance, and G is called the *conclusion* of the $\rhd$-instance. A $\rhd$*-derivation* of a sentence $G \in \mathfrak{S}^{\mathrm{par}}$ from a set $S \subset \mathfrak{S}^{\mathrm{par}}$ of sentences is a tuple $\langle F_0, \dots, F_N \rangle$ of sentences of $\mathfrak{S}^{\mathrm{par}}$ such that $F_N = F$ and each F_i (with $i = 0, \dots, N$) is an element of S or the conclusion of a $\rhd$-instance whose premises are among the sentences $F_0, \dots, F_{i-1}$. A $\rhd$*-proof* of a sentence $F \in \mathfrak{S}^{\mathrm{par}}$ is a $\rhd$-derivation of F from the empty set. If $S \subset \mathfrak{S}^{\mathrm{par}}$ and $E \in \mathfrak{S}^{\mathrm{par}}$ then $S \vdash_{\rhd} E$ means that there is a $\rhd$-derivation of E from S.

The set of instances of a rule of a Frege-Hilbert calculus is a deduction relation. Likewise, the set of rule instances of a Frege-Hilbert calculus is a deduction relation. By a *derivation* of F from S in a Frege-Hilbert calculus we mean a derivation of F from S with respect to this deduction relation, provided that F and the sentences of S have no parameters.

Rules of Frege-Hilbert calculi usually have a fixed arity, i.e., number of premises. Similarly, we say that a deduction relation $\rhd \subset \mathfrak{S}^{\mathrm{par}\,n} \times \mathfrak{S}^{\mathrm{par}}$ has *arity* n.

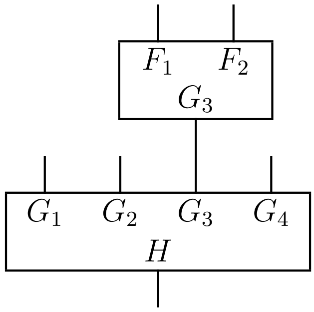
4. Composition and factorization of deduction relations

On the set of deduction relations which have an arity, the two operations of composition and factorization are defined as follows.

For $i = 1, \dots, n$, the i*-th composition* $\rhd_1 i \rhd_2$ of two relations $\rhd_1 \subset \mathfrak{S}^{\mathrm{par}\,m} \times \mathfrak{S}^{\mathrm{par}}$ and $\rhd_2 \subset \mathfrak{S}^{\mathrm{par}\,n} \times \mathfrak{S}^{\mathrm{par}}$ is defined by

$$\begin{aligned} \rhd_1 i \rhd_2 \;=\; \{ \langle \langle G_1, \dots, G_{i-1}, F_1, \dots, F_m, G_{i+1}, \dots, G_n \rangle, H \rangle \mid \\ \langle F_1, \dots, F_m \rangle \rhd_1 G_i \text{ and } \langle G_1, \dots, G_n \rangle \rhd_2 H \} \end{aligned}$$

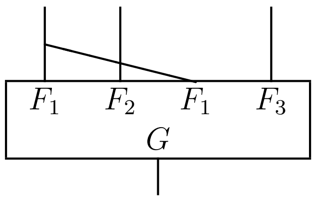
Here is a picture for $m = 2$, $n = 4$, and $i = 3$. It shows an instance $\langle \langle F_1, F_2 \rangle, G_3 \rangle$ of $\rhd_1$ and an instance $\langle \langle G_1, G_2, G_3, G_4 \rangle, H \rangle$ of $\rhd_2$. Each of these instances is shown as a box with the premises written below its upper edge and the conclusion written above its lower edge. The picture looks similar to an electric circuit and, in fact, the composition of deduction relations can be depicted as plugging together two boxes. The free ends represent the instance $\langle \langle G_1, G_2, F_1, F_2, G_4 \rangle, H \rangle$ of the composition $\rhd_1 i \rhd_2$.



If $K_1, \dots, K_k$ are the equivalence classes of $\{1, \dots, n\}$ with respect to some equivalence relation, then the $\langle K_1, \dots, K_k \rangle$*-factor* of an n-ary deduction relation $\rhd \subset \mathfrak{S}^{\mathrm{par}\,n} \times \mathfrak{S}^{\mathrm{par}}$ is the k-ary deduction relation $\rhd' \subset \mathfrak{S}^{\mathrm{par}\,k} \times \mathfrak{S}^{\mathrm{par}}$ defined by

$$\langle F_1, \dots, F_k \rangle \rhd' G \;:\Longleftrightarrow\; \langle F_{j_1}, \dots, F_{j_n} \rangle \rhd G$$

where j_i is defined by $i \in K_{j_i}$ for $i = 1, \dots, n$. We say that $\rhd'$ has been obtained from $\rhd$ by *factorization*. Here is a picture for $n = 4$, $k = 3$, $K_1 = \{1, 3\}$, $K_2 = \{2\}$, and $K_3 = \{4\}$. In this example, $\langle F_1, F_2, F_3 \rangle \rhd' G$ holds if and only if $\langle F_1, F_2, F_1, F_3 \rangle \rhd G$ holds. Factorization can be depicted as wiring together some of the premises. The free ends represent the instance $\langle \langle F_1, F_2, F_3 \rangle, G \rangle$ of the factor $\rhd'$.

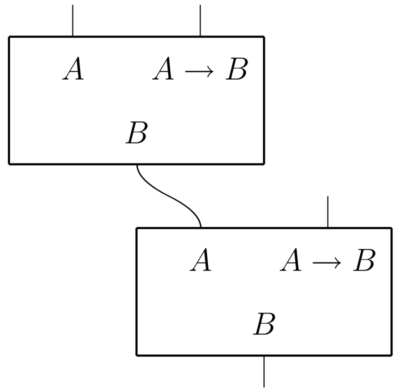


If D is a set of deduction relations with arities then the *closure* $C_{\mathrm{cf}}(D)$ of D with respect to composition (c) and factorization (f) is the smallest set of deduction relations which contains all deduction relations of D and is closed under composition and factorization. The sets $C_{\mathrm{c}}(D)$ and $C_{\mathrm{f}}(D)$ are defined similarly.

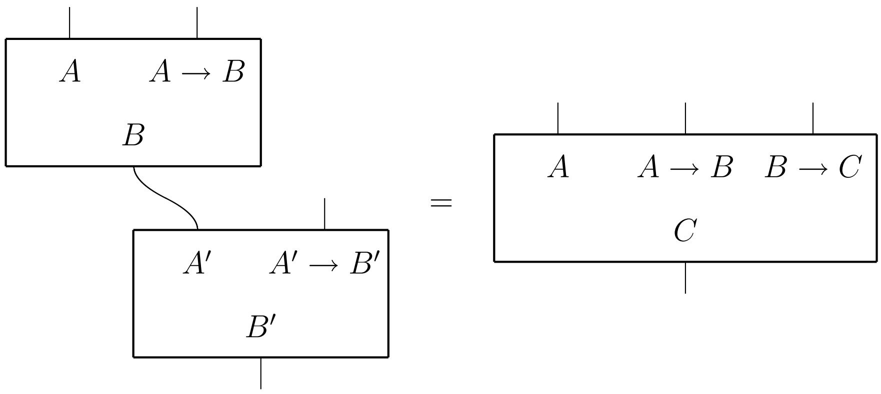
Now, as was already said above, each rule of a Frege-Hilbert calculus $\mathcal{C}$ defines a deduction relation. Let $D_{\mathcal{C}}$ be the set of all these deduction relations. Then the set of all deduction relations obtained by combining applications of rules of the Frege-Hilbert calculus and by possibly identifying some of the premises with each other, is exactly the set $C_{\mathrm{cf}}(D_{\mathcal{C}})$. So, if we make a proof attempt and start to somehow combine rules of the calculus without knowing yet exactly which formulas we should take, then the result can be described by a deduction relation in $C_{\mathrm{cf}}(D_{\mathcal{C}})$. So, the next goal is to learn more about composition and factorization of deduction relations defined by rules of Frege-Hilbert calculi, and to determine the closure $C_{\mathrm{cf}}(D_{\mathcal{C}})$. It is, however, difficult to do computation with deduction relations, since deduction relations are infinite sets. Therefore, deduction relations should be represented by finite structures. We know already that rules can represent deduction relations. So, we should define operations of composition and factorization on rules.

5. Composition, factorization, and partial instantiation of rules

Rules of Frege-Hilbert calculi can be composed with each other as has been shown in [5]. As a simple example, let us consider the modus ponens rule (R1). The first composition (R1)1(R1) of (R1) with itself can be obtained in analogy to the composition of deduction relations. Let us draw a picture.



Here, "A" and "B" are meta-symbols for formulas. They are local to each of the boxes. So, for example, "A" in the upper box in general does not refer to the same formula as "A" in the lower box. We should better rename the meta-symbols in the lower box. Then we get



As we see, the composition of rules involves a unification, not on terms but on formulas. Composition of rules for quantification is a lot more complex and involves constraints, in particular in form of equations between formulas, in addition to simple unification. Details can be found in [5].

Factorization of rules is rather similar since it is also done by unification of formula schemes containing meta-symbols. Here, however the premises of a rule are unified with each other. But the problems and the ways to solve them are the same as for the composition of rules.

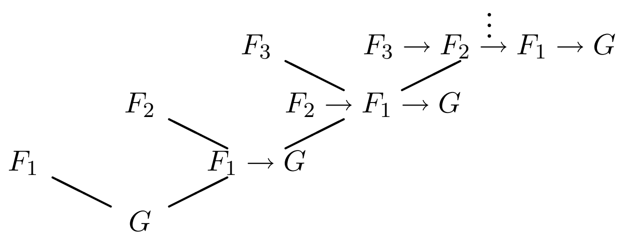
It follows that the operations on deduction relations which we are interested in, can be carried out explicitly and automatically on the rule representations of deduction relations. Moreover, the deduction relations in $C_{\text{cf}}(D_{\mathcal{C}})$ actually have representations as rules. However, these rules are more complex than those Frege-Hilbert rules which we are used to from literature.

In addition to the operations of composition and factorization, the set of rules has the operation of *partial instantiation*. A rule R' is a *partial instance* of a rule R if R' is obtained from R by consistently replacing meta-symbols for formulas and for terms with formula schemes or term schemes, respectively, and by injectively replacing meta-symbols for variables and for parameters with meta-symbols for variables, or parameters, respectively. For example, the rule $\frac{\neg F \quad \neg F \to G \wedge H}{G \wedge H}$ is a partial instance of the modus ponens rule (R1), since it is obtained from (R1) by consistently replacing "A" with "$\neg F$" and "B" with "$G \wedge H$". Every instance of a partial instance of a rule R is also an instance of R. Therefore, a partial instance of a rule R is a weaker rule than R. As for deduction relations, closures with respect to composition, factorization, but also with respect to partial instantiation, are defined for sets of rules. In particular, for a Frege-Hilbert calculus $\mathcal{C}$, the closure $C_{\text{cfi}}(\mathcal{C})$ is the smallest set of rules which contains all rules of $\mathcal{C}$ and which is closed under composition (c), factorization (f), and partial instantiation (i). It can be shown that the deduction relation defined by $C_{\mathrm{cf}}(\mathcal{C})$ is the closure of the deduction relation defined by $\mathcal{C}$: It holds $\rhd_{C_{\mathrm{cf}}(\mathcal{C})} = C_{\mathrm{cf}}(\rhd_{\mathcal{C}})$ and likewise for C_{c} and C_{f}.

6. Figures

A *figure* has the form $\frac{F_1 \quad \cdots \quad F_n}{G}$ where $F_1, \ldots, F_n, G$ are closed formulas. Letting the function symbols, predicate symbols, variables, and parameters of $F_1, \ldots, F_n, G$ play the role of meta-variables, a figure can be considered as a rule. The advantage of this approach is that we can use concepts like validity and semantic entailment for the formulas $F_1, \ldots, F_n, G$. Also, we can make use of well-known theorems such as the deduction theorem to prove properties of Frege-Hilbert calculi. The rules of the Frege-Hilbert calculi known from literature can be presented as figures. A disadvantage of figures is that there seems to be no obvious way to represent a composition of two figures as a figure in first order logic. Either constraints have to be added to the figures or a composition of two figures must be represented as a possibly infinite set of figures.

Yet, a number of interesting results can be obtained from studying figures. For example, if $\mathcal{C}$ is a sound and complete Frege-Hilbert calculus of propositional logic which contains the modus ponens rule (R1) then all valid rules can be obtained just by composition and partial instantiation of rules of $\mathcal{C}$. Factorization is not necessary. More precisely, $C_{\mathrm{ci}}(\mathcal{C}) = C_{\mathrm{cfi}}(\mathcal{C})$. The idea of the proof is that a rule R of $C_{\mathrm{cfi}}(\mathcal{C})$ can be presented as a figure $\frac{F_1 \quad \cdots \quad F_k}{G}$. Then $F_k \to \ldots \to F_1 \to G$ is a valid formula and has therefore a derivation in $\mathcal{C}$. Let R_0 be the composition of rules of $\mathcal{C}$ corresponding to this derivation. Then R is a partial instance of the rule $(\ldots((\mathrm{R}_0 2(\text{R1}))2(\text{R1}))\ldots 2(\text{R1}))$.



If, however, (R1) is replaced by $\frac{A \quad A \quad A \to B \quad A \to B}{B}$ then factorization is necessary to obtain all valid rules from the rules of the calculus. Here is another interesting result for propositional logic:

1. A derivation of a valid formula in one adequate Frege-Hilbert calculus $\mathcal{C}$ can be simulated at linear cost in any other Frege-Hilbert calculus $\mathcal{C}'$.
2. If (R1) $\in C_{\mathrm{c}}(\mathcal{C}')$ then this holds also for tree derivations.

7. Future research

Among the tasks and questions to be addressed in future research are the following. Lift some of the results known for propositional logic to the first order and classify calculi according to which operations are necessary to generate all valid rules from their rules. Is partial instantiation necessary to generate all valid rules? How can one Frege-Hilbert calculus be simulated by another one?

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