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A Bifurcation Model of Quantum State Reduction on the Unit Circle: a nonlinear unitary extension of the Schrödinger equation

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In a two-dimensional Cartesian space I first consider a point P with coordinates (x, y) moving on S^1 s.t. $|x(t)|^2 + |y(t)|^2 = 1, \ \forall t \in [0, \tau]$, where τ is some time lapse, together with initial data $\{x(0), y(0)\}$. A certain nonlinear dynamics is established, along with suitable initial conditions, s.t. the angular speed of $\mathbf{P}$ depends explicitly on (x, y): the *evolution* of $\mathbf{P}$ then bifurcates on S^1 at $t = 0$, with $xy = 0$ at $t = \tau$.

Although no example is known in classical mechanics, it is inferred that such a nonlinear model quite appropriately provides a framework for *quantum state reduction* in a two-dimensional Hilbert space: the point $\mathbf{P}$ is then replaced by a *quantum state* (of unit l^2-norm), wherein the Cartesian data $\{|x(0)|^2, |y(0)|^2\}$ are to express the associated *quantum probabilities* for mutually exclusive outcomes.

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1. A nonlinear classical model on the unit circle

In Cartesian space I first consider a point $\mathbf{P}$ with coordinates (x, y), and parametrization:

$$x = \cos\phi(t), \quad y = \sin\phi(t) \ ,$$

where ϕ depends on the time-variable t, s.t. there exists a vector

$$V = \begin{pmatrix} x \\ y \end{pmatrix} \text{, with derivative } \frac{dV}{dt} = \mathbf{M}V \ ; \tag{1}$$

here $\mathbf{M}$ is a 2x2-matrix, given by $\omega \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \ \omega = \frac{d\phi}{dt}$.

Let $\mathbf{M}^T = -\mathbf{M}$ denote the transpose matrix, then on $S^1 : V^T V = 1, \ \forall t \in [0, \tau]$, where τ is some given time lapse.

Remark: Note that $\frac{dV}{dt}$ is linear in V if the *angular speed* ω does not depend on $\mathbf{P}(x, y)$. Indeed, this model is nonlinear if ω depends explicitly on $\mathbf{P}(x, y)$, for example [1]:

$$\omega = -\frac{g_0}{(x^2 - y^2)}, \ t \in (0, \tau) \ ,$$

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where g_0 is, for the sake of simplicity, some real valued constant; by assuming the following *initial condition* $\int_0^\tau g_0\, dt = x(0)y(0)$, then :

$$\phi = \frac{1}{2}\sin^{-1}\left(2\int_t^\tau g_0\, dt\right),\ \forall t \in [0,\tau];$$

incidentally, one could choose, instead of g_0, a more general function $g(x, y, t)$.

Let us operate within the first quadrant, then ϕ is either 0 or $\frac{\pi}{2}$ at $t = \tau$, i.e. the initial point with coordinates $\{x(0), y(0)\}$ is a bifurcation-point: for $t > 0$ the evolution of $\mathbf{P}$ is then *determined* by the nonlinear differential equation

$$\frac{dV}{dt} = -\frac{g_0}{(x^2 - y^2)}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} V, \tag{2}$$

yet V is not computable, i.e.: the point $\mathbf{P} \in S^1$ is developing *parallel histories*, one history ending at $\phi = 0$ and the other at $\phi = \frac{\pi}{2}$.

Remark: No such evolution is known in classical physics, where one associates with $\mathbf{P}$ a *massive particle* abiding with Newtonian mechanics. However, development of *parallel histories* has come about in quantum mechanics within the realm of *quantum state reduction* (see ref. [1] and the references therein); then τ would be the *collapsing time* for this reduction!

2. A nonlinear unitary quantum model of reduction

Transition to Hilbert space: Let us associate with $\mathbf{P}$ a quantum state, $|p\rangle$, say (replacing the above Cartesian vector V) in a two-dimensional Hilbert space, s.t. [1]

$$i\frac{d\,|p\rangle}{dt} = \mathbf{R}\,|p\rangle\,,\ t \in (0,\tau), \tag{3}$$

where $R = \omega\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, being a Hermitian matrix (replacing the above $\mathbf{M}$); then there exists a *dual* description

$$-i\frac{d\,\langle p|}{dt} = \langle p|\,\mathbf{R},\ t \in (0,\tau)$$

with the choice $\langle p|p\rangle = 1,\ \forall t \in [0,\tau]$ (replacing the above Cartesian norm $V^T V = 1$).

Note that this model, eq.(3), admits the following formal solution:

$$|p\rangle = \begin{pmatrix} \xi \\ \eta \end{pmatrix},\ \text{s.t. for \textit{real amplitudes}: } \xi = \cos\phi,\ \eta = \sin\phi,$$

where ϕ is time-dependent, with $\omega = \frac{d\phi}{dt}$; then, for example, eq.(2) is replaced by the nonlinear *evolution* equation

$$i\frac{d\,|p\rangle}{dt} = -\frac{g_0}{(\xi^2 - \eta^2)}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}|p\rangle\,,\ \text{with } \int_0^\tau g_0\, dt = \xi(0)\eta(0); \tag{4}$$

this integral-condition on $[0, \tau]$ ensures that suitable detectors will select a single outcome from a set of two alternatives (see below, eq.(6)). Also note that a generalization to *complex amplitudes* (accounting for interference-effects: see appendix) is not difficult, i.e. [1]

$$i\frac{d\,|p\rangle}{dt} = -\frac{1}{|\xi|^2 - |\eta|^2}\begin{pmatrix} 0 & -i\,g_0 \\ i\,\overline{g_0} & 0 \end{pmatrix}|p\rangle\,,\ t \in (0,\tau), \text{ with } \int_0^\tau g_0\,dt = \xi(0)\overline{\eta(0)}$$

where g_0 is a complex valued constant. This equation can be made even more general if g_0 is replaced by some appropriate complex function $g(\xi, \eta, t)$.

Remark: Notice that eq.(3) bears resemblance to the *Schrödinger* equation

$$i\frac{d\,|p\rangle}{dt} = \mathbf{H}\,|p\rangle \tag{5}$$

where the *Hamiltonian* $\mathbf{H}$ is a linear Hermitian operator; however, this fundamental equation of quantum theory only holds prior to measurement, for $t \in (-\infty, 0)$, say.

A simple example: Consider a photon beam incident on a semi-transparent mirror, this beam splits up into a reflected ray and a second one (of equal amplitude) that passes through the mirror; the overall quantum state, denoted by $|p\rangle$, will satisfy eq.(5) with a Hamiltonian given by

$$\mathbf{H} = 2\pi\nu\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \text{ where } \nu \text{ is the frequency of the beam.}$$

The solution of eq.(5) is then given by

$$|p\rangle = e^{-2\pi i\nu t}\begin{pmatrix} \cos\phi_0 \\ \sin\phi_0 \end{pmatrix}, \text{ with } \phi_0 = \frac{\pi}{4},\ \forall t \in (-\infty, 0],$$

s.t. in this case the quantum probabilities for detecting a photon are the same, i.e. equal to $1/2$.

Now, as to the problem of reduction, we employ eq.(3) and eq.(4), s.t. at $t = \tau$ (also see [1]):

$$|p\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \text{or} \quad |p\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{6}$$

i.e.: the final state is not *computable*.

Probability and bifurcation: When one tosses a *fair* coin the probability for either outcome (heads or tails) is 50%. In fact, according to Newtonian mechanics, if we would know all the physical details at some given time, we could actually compute the outcome in advance. However, no such computation for mutually exclusive outcomes is logically possible in quantum theory: Principally, there is a lack of *additional information* - as is the case when we try to compute a *unique number* ϕ from a transcendental equation, like (see above)

$$\phi = \frac{1}{2}\sin^{-1}\Omega(t),\ (0 \le t \le \tau), \text{ at } t = \tau, \text{ s.t. } \Omega(\tau) = 0.$$

(there is no such single number at $t = \tau$, even though this might be a solution of a *deterministic evolution* equation !)

So, it appears that the concept of *quantum probability* is ultimately related to that of *bifurcation* in the present sense (see appendix in [1]).

Interferometry: According to the figure below, in a Mach-Zehnder interferometer [2] an incident beam of photons is split by a splitter B (as above), and then reflected by a pair of equidistant mirrors (M_1, M_2) onto a second identical beam splitter B' (diagonally located and installed parallel to B). At the location of B' we get a new state $\begin{pmatrix} \cos\phi_0' \\ \sin\phi_0' \end{pmatrix}$, where $\phi_0' = 2\phi_0$ (all we need to know is, for $\phi_0 = 0$ there is 100% reflection at B', and for $\phi = \frac{\pi}{2}$ there is 100% transmission at B').

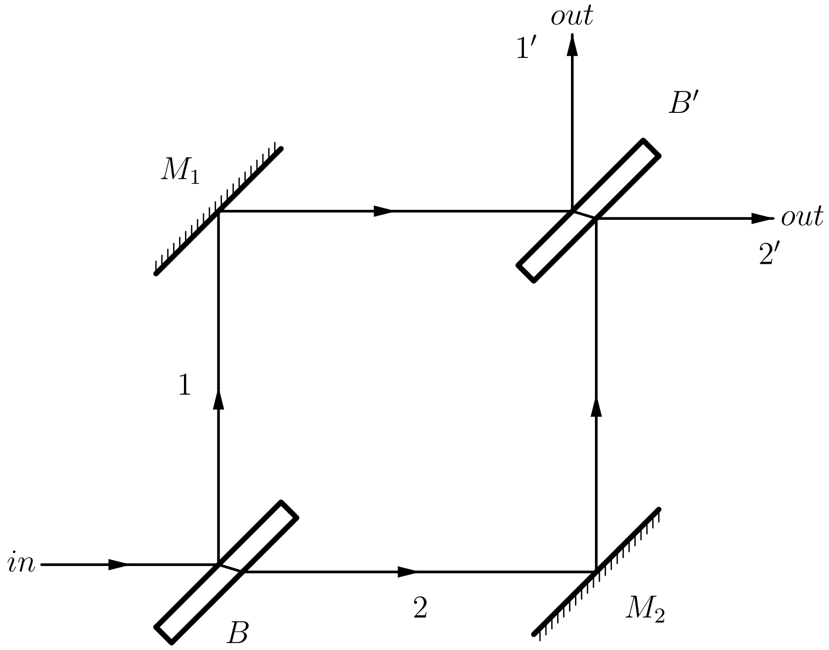

Figure 1. Illustrating interference in the Mach-Zehnder Interferometer.

Here the meaning of complex amplitudes becomes apparent: Call $|p\rangle$ the incoming state, then at the first splitter B we simply write down a linear superposition, spanned by orthonormal base-vactors:

$$|p\rangle \quad \rightarrow \quad \cos\phi_0\,|1\rangle + \sin\phi_0\,|2\rangle\;;$$

however, at the second splitter B' we ought to introduce complex amplitudes (with primed base-vectors):

$$\begin{aligned} |1\rangle \quad &\rightarrow \quad e^{i\alpha}\cos\phi_0\,|1'\rangle + e^{i\beta}\sin\phi_0\,|2'\rangle\;, \\ |2\rangle \quad &\rightarrow \quad e^{i\gamma}\sin\phi_0\,|1'\rangle + e^{i\delta}\cos\phi_0\,|2'\rangle\;; \end{aligned}$$

for the sake of simplicity, put $\alpha = \beta = \delta = 0$ also observing $\langle 1|2\rangle = 0$, with $\gamma = \pi$ (indicating destructive interference in the direction of $1'$, with the probability amplitude also depending on ϕ_o), then the desired result follows:

$$|p\rangle \quad \rightarrow \quad \cos 2\phi_0\,|1'\rangle + \sin 2\phi_0\,|2'\rangle\;;$$

thus, when these beam splitters are semi-transparent ($\phi_0 = \frac{\pi}{4}$), then the final state of a photon, leaving B', is exactly $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, i.e. photons leave the splitter B' in the same direction they came into the apparatus (from a different viewpoint, this particular scenario has also been discussed in [3], [4] and recently again in [5], with the same result):

Due to *interference* of the split beam at B', we cannot compute the path of a photon after it had passed the first beam splitter B; so, when detectors are suitably placed between B and B' we expect the *initial* state of an incoming photon (incident on B) to *collapse* unpredictably, according to eq.(6), say.

Final Remark: By making use of a rather simple bifurcation scheme, this paper expresses a unifying aspect within different areas of quantum mechanics through a Schrödinger-type evolution equation [1]

$$i\frac{d\,|p\rangle}{dt} = \mathbf{Q}\,|p\rangle$$

here $\mathbf{Q}$ is either $\mathbf{H}$ (the linear Schrödinger-Hamiltonian), or $\mathbf{R}$ (the nonlinear *reduction operator*), according to whether $t \in (-\infty, 0)$, or $t \in (0, \tau)$, say; eq.(5) only holds up to the time of measurement which sets about at $t = 0$, whereas eq.(3) relates to the actual *measurement process* which is completed at $t = \tau$.

Note that we are dealing with different *elements of reality*: There are photons at the quantum-level, and secondly there are macroscopic gadgets (beam splitters and detectors); so we expect that τ depends on ν, as well as on the *admissible size* of these devices [1]. I also suspect that for $\nu > \nu_0$, where ν_0 is some cutoff, there is no coherent splitting into rays of comparable amplitudes; for frequencies beyond this cutoff I would not expect that quantum-mechanical interference (as in interferometry) can be resolved by any kind of experiment. Also note that there are various views on the *dividing line* between *micro-and macro* objects as discussed in [6] and very recently in [7].

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